

Time-Periodic Second-Order Hyperbolic Equations: Fredholmness, Regularity, and Smooth Dependence

I. Kmit L. Recke

Institute of Mathematics, Humboldt University of Berlin,
Rudower Chaussee 25, D-12489 Berlin, Germany
and Institute for Applied Problems of Mechanics and Mathematics,
Ukrainian Academy of Sciences, Naukova St. 3b, 79060 Lviv, Ukraine
E-mail: kmit@informatik.hu-berlin.de

Institute of Mathematics, Humboldt University of Berlin,
Rudower Chaussee 25, D-12489 Berlin, Germany
E-mail: recke@mathematik.hu-berlin.de

Abstract

The paper concerns the general linear one-dimensional second-order hyperbolic equation

$$\partial_t^2 u - a^2(x, t) \partial_x^2 u + a_1(x, t) \partial_t u + a_2(x, t) \partial_x u + a_3(x, t) u = f(x, t), \quad x \in (0, 1)$$

with periodic conditions in time and Robin boundary conditions in space. Under a non-resonance condition (formulated in terms of the coefficients a , a_1 , and a_2) ruling out the small divisors effect, we prove the Fredholm alternative. Moreover, we show that the solutions have higher regularity if the data have higher regularity and if additional non-resonance conditions are fulfilled. Finally, we state a result about smooth dependence on the data, where perturbations of the coefficient a lead to the known loss of smoothness while perturbations of the coefficients a_1 , a_2 , and a_3 do not.

Key words: second-order hyperbolic equation, periodic conditions in time, Robin conditions in space, non-resonance conditions, Fredholm alternative, regularity of solutions, smooth dependence on the data

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1 Introduction

1.1 Problem setting and main results

We address the questions of Fredholm solvability, regularity of solutions and smooth dependence on the data for the general linear one-dimensional second-order hyperbolic equation

$$\partial_t^2 w - a^2(x, t) \partial_x^2 w + a_1(x, t) \partial_t w + a_2(x, t) \partial_x w + a_3(x, t) w = f(x, t), \quad x \in (0, 1) \quad (1)$$

subjected to periodic conditions in time

$$w(x, t) = w(x, t + T), \quad x \in (0, 1), \quad (2)$$

and Robin boundary conditions in space

$$\begin{aligned} \partial_x w(0, t) &= r_0(t) w(0, t), \\ \partial_x w(1, t) &= r_1(t) w(1, t). \end{aligned} \quad (3)$$

Here $T > 0$ is a fixed real number. The functions $a, a_1, a_2, a_3, f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $r_0, r_1 : \mathbb{R} \rightarrow \mathbb{R}$ are supposed to be T -periodic with respect to t and to satisfy

$$a(x, t) > 0 \text{ for all } x \in [0, 1] \text{ and } t \in \mathbb{R} \quad (4)$$

and $\int_0^T a(0, t) r_0(t) dt \neq 0$ or $\int_0^T a(1, t) r_1(t) dt \neq 0$. Without loss of generality, throughout the paper we will assume that

$$\int_0^T a(0, t) r_0(t) dt \neq 0. \quad (5)$$

We will simply write C_T^l for the Banach space of T -periodic in t and l -times continuously differentiable functions $u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, with the usual norm

$$\|u\|_\infty + \sum_{i=1}^l (\|\partial_x^i u\|_\infty + \|\partial_t^i u\|_\infty), \quad (6)$$

where

$$\|u\|_\infty = \max_{0 \leq x \leq 1} \max_{t \in \mathbb{R}} |u(x, t)|. \quad (7)$$

Moreover, we let $\mathcal{C}^l = C_T^l \times C_T^l$. The norm in \mathcal{C}^l is again given by (6)–(7) but $|\cdot|$ in (7) is now used to denote the Euclidean norm in \mathbb{R}^2 . Also, by $C_T^l(\mathbb{R})$ we will denote the Banach space of T -periodic and l -times continuously differentiable functions $u : \mathbb{R} \rightarrow \mathbb{R}$. Similarly, let C_T^∞ (resp., $C_T^\infty(\mathbb{R})$) denote the space of T -periodic in t and infinity differentiable functions $u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ (resp., $u : \mathbb{R} \rightarrow \mathbb{R}$).

The problem (1)–(3) can be written as a problem for a first-order hyperbolic integro-differential system. Indeed, set $u = (u_1, u_2)$ and

$$C = \int_0^T a(0, t) r_0(t) dt \quad (8)$$

and introduce linear bounded operators $N : \mathcal{C} \mapsto \mathbb{R}$, $I, G : \mathcal{C} \mapsto C_T(\mathbb{R})$ and $J, F : \mathcal{C} \mapsto \mathcal{C}$ by

$$[Iu](t) = \int_0^t \frac{u_1(0, \tau) + u_2(0, \tau)}{2} d\tau, \quad (9)$$

$$[Ju](x, t) = \int_0^x \frac{u_1(\xi, t) - u_2(\xi, t)}{2a(\xi, t)} d\xi, \quad (10)$$

$$Nu = \frac{1}{C} \int_0^T \left(\frac{u_1(0, t) - u_2(0, t)}{2} - a(0, t)r_0(t)[Iu](t) \right) dt, \quad (11)$$

$$[Gu](t) = [Iu](t) + Nu, \quad (12)$$

$$[Fu](x, t) = [Gu](t) + [Ju](x, t). \quad (13)$$

Moreover, we introduce the following notation:

$$\begin{aligned} b_{11} &= \frac{a_1}{2} + \frac{a_2}{2a} + \frac{a\partial_x a - \partial_t a}{2a}, & b_{12} &= \frac{a_1}{2} - \frac{a_2}{2a} + \frac{a\partial_x a - \partial_t a}{2a}, \\ b_{21} &= \frac{a_1}{2} + \frac{a_2}{2a} + \frac{a\partial_x a + \partial_t a}{2a}, & b_{22} &= \frac{a_1}{2} - \frac{a_2}{2a} - \frac{a\partial_x a + \partial_t a}{2a}. \end{aligned} \quad (14)$$

In the new unknowns

$$u_1 = \partial_t w + a(x, t)\partial_x w, \quad u_2 = \partial_t w - a(x, t)\partial_x w \quad (15)$$

the problem (1)–(3) reads as follows:

$$\begin{aligned} \partial_t u_1 - a(x, t)\partial_x u_1 + b_{11}(x, t)u_1 + b_{12}(x, t)u_2 &= f(x, t) - [a_3 Fu](x, t) \\ \partial_t u_2 + a(x, t)\partial_x u_2 + b_{21}(x, t)u_1 + b_{22}(x, t)u_2 &= f(x, t) - [a_3 Fu](x, t), \end{aligned} \quad (16)$$

$$u_j(x, t) = u_j(x, t + T), \quad j = 1, 2, \quad (17)$$

$$\begin{aligned} u_1(0, t) &= u_2(0, t) + 2a(0, t)r_0(t)[Gu](t), \\ u_2(1, t) &= u_1(1, t) - 2a(1, t)r_1(t)[Fu](1, t). \end{aligned} \quad (18)$$

It is not difficult to check (see Section 2) that the problems (1)–(3) and (16)–(18) are equivalent in the sense of the classical solvability, namely, that any classical solution to (1)–(3) produces a classical solution to (16)–(18) by means of the formula (15) and, vice versa, any classical solution to (16)–(18) produces a classical solution to (1)–(3) by means of the formula

$$w(x, t) = [Iu](t) + [Ju](x, t) + Nu. \quad (19)$$

We will work with the concepts of a weak (continuously differentiable) solution to (1)–(3) and a weak (continuous) solution to (16)–(18), based on the integration along characteristics. In order to switch to the weak formulations, let us introduce characteristics of the system (16). Given $j = 1, 2$, $x \in [0, 1]$, and $t \in \mathbb{R}$, the j -th characteristic is defined as the solution $\xi \in [0, 1] \mapsto \tau_j(\xi, x, t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \tau_j(\xi, x, t) = \frac{(-1)^j}{a(\xi, \tau_j(\xi, x, t))}, \quad \tau_j(x, x, t) = t. \quad (20)$$

In what follows we will write

$$c_j(\xi, x, t) = \exp \int_x^\xi (-1)^j \left(\frac{b_{jj}}{a} \right) (\eta, \tau_j(\eta, x, t)) d\eta, \quad (21)$$

$$d_j(\xi, x, t) = \frac{(-1)^j c_j(\xi, x, t)}{a(\xi, \tau_j(\xi, x, t))}. \quad (22)$$

Due to the method of characteristics, a C^1 -map $u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^2$ is a solution to the problem (16)–(18) if and only if it satisfies the following system of integral equations

$$\begin{aligned} u_1(x, t) = & c_1(0, x, t) \left[u_2(0, \tau_1(0, x, t)) \right. \\ & + 2a(0, \tau_1(0, x, t)) r_0(\tau_1(0, x, t)) [Gu](\tau_1(0, x, t)) \Big] \\ & - \int_0^x d_1(\xi, x, t) b_{12}(\xi, \tau_1(\xi, x, t)) u_2(\xi, \tau_1(\xi, x, t)) d\xi \\ & + \int_0^x d_1(\xi, x, t) \left(f(\xi, \tau_1(\xi, x, t)) - [a_3 Fu](\xi, \tau_1(\xi, x, t)) \right) d\xi, \end{aligned} \quad (23)$$

$$\begin{aligned} u_2(x, t) = & c_2(1, x, t) \left[u_1(1, \tau_2(1, x, t)) \right. \\ & - 2a(1, \tau_2(1, x, t)) r_1(\tau_2(1, x, t)) [Fu](1, \tau_2(1, x, t)) \Big] \\ & - \int_1^x d_2(\xi, x, t) b_{21}(\xi, \tau_2(\xi, x, t)) u_1(\xi, \tau_2(\xi, x, t)) d\xi \\ & + \int_1^x d_2(\xi, x, t) \left(f(\xi, \tau_2(\xi, x, t)) - [a_3 Fu](\xi, \tau_2(\xi, x, t)) \right) d\xi. \end{aligned} \quad (24)$$

As it follows from Section 2, if (u_1, u_2) is a continuous vector-function satisfying the system (23)–(24), then the function w given by (19) is continuously differentiable. Hence, the notion of a weak solution to the problem (1)–(3) can be naturally defined as follows:

Definition 1.1 (i) A function $u \in \mathcal{C}$ is called a weak solution to (16)–(18) if it satisfies (23) and (24).

(ii) Let u be a weak solution to (16)–(18). Then the continuously differentiable function w given by the formula (19) is called a weak solution to (1)–(3).

Denote by \mathcal{K}_w the vector space of all weak solutions to (1)–(3) with $f = 0$. We are prepared to state the Fredholm alternative theorem.

Theorem 1.2 Suppose (4) and (5). Moreover, assume that

$$a \in C_T^2, \quad a_1, a_2 \in C_T^1, \quad a_3 \in C_T, \quad r_0, r_1 \in C_T^1(\mathbb{R}) \quad (25)$$

and either

$$\int_0^1 \left[\left(\frac{b_{11}}{a} \right) (\eta, \tau_1(\eta, 1, t)) + \left(\frac{b_{22}}{a} \right) (\eta, \tau_2(\eta, 0, \tau_1(0, 1, t))) \right] d\eta \neq 0 \quad (26)$$

or

$$\int_0^1 \left[\left(\frac{b_{22}}{a} \right) (\eta, \tau_2(\eta, 0, t)) + \left(\frac{b_{11}}{a} \right) (\eta, \tau_1(\eta, 1, \tau_2(0, 1, t))) \right] d\eta \neq 0 \quad (27)$$

for all t . Then the following is true:

(i) $\dim \mathcal{K}_w < \infty$.

(ii) The space of all $f \in C_T$ such that there exists a weak solution to (1)–(3) is a closed subspace of codimension $\dim \mathcal{K}_w$ in C_T .

(iii) Either $\dim \mathcal{K}_w > 0$ or for any $f \in C_T$ there exists exactly one weak solution w to (1)–(3).

Remark 1.3 It follows from non-resonance conditions (26) and (27) that, in general, resonances are defined by coefficients a , a_1 , and a_2 of the second- and the full first-order part of the equation (1). In the particular case $\partial_t a \equiv 0$ conditions (26) and (27) can be written in the form

$$\int_0^1 \frac{b_{11}(\eta) + b_{22}(\eta)}{a(\eta)} d\eta = \int_0^1 \frac{a(\eta)a_1(\eta) + a(\eta)a'(\eta)}{a^2(\eta)} d\eta \neq 0. \quad (28)$$

This means that in this case the resonances do not depend on the coefficient a_2 any more.

To formulate a regularity result, we introduce the notation

$$c_j^l(\xi, x, t) = \exp \int_x^\xi (-1)^j \left(\frac{b_{jj}}{a} - l \frac{\partial_t a}{a^2} \right) (\eta, \tau_j(\eta, x, t)) d\eta.$$

Notice that $c_j^0(\xi, x, t) = c_j(\xi, x, t)$.

Theorem 1.4 Suppose (4) and (5).

(i) Given $k \geq 1$, assume that

$$a \in C_T^{k+1}, \quad a_1, a_2, a_3, f \in C_T^k, \quad r_0, r_1 \in C_T^k(\mathbb{R}) \quad (29)$$

and one of the following conditions is fulfilled:

$$c_1^l(0, 1, t)c_2^l(1, 0, \tau_1(0, 1, t)) < 1 \text{ for all } t \in \mathbb{R} \text{ and } l = 0, 1, \dots, k, \quad (30)$$

$$c_1^l(0, 1, t)c_2^l(1, 0, \tau_1(0, 1, t)) > 1 \text{ for all } t \in \mathbb{R} \text{ and } l = 0, 1, \dots, k, \quad (31)$$

$$c_2^l(1, 0, t)c_1^l(0, 1, \tau_2(1, 0, t)) < 1 \text{ for all } t \in \mathbb{R} \text{ and } l = 0, 1, \dots, k, \quad (32)$$

and

$$c_2^l(1, 0, t)c_1^l(0, 1, \tau_2(1, 0, t)) > 1 \text{ for all } t \in \mathbb{R} \text{ and } l = 0, 1, \dots, k. \quad (33)$$

Then any weak solution to (1)–(3) belongs to C_T^{k+1} .

(ii) Assume that a is independent of t . Moreover, let $a, a_1, a_2, a_3, f \in C_T^\infty$ and $r_0, r_1 \in C_T^\infty(\mathbb{R})$. If one of the conditions (26) and (27) is fulfilled, then any weak solution to (1)–(3) belongs to C_T^∞ .

We finish this section with the theorem describing smooth dependence of the solutions on the data. With this aim, by means of $\varepsilon \in [0, 1)$, we perform small perturbations

$$\begin{aligned} a^\varepsilon(x, t) &= a(x, t, \varepsilon), & a_1^\varepsilon(x, t) &= a_1(x, t, \varepsilon), & a_2^\varepsilon(x, t) &= a_2(x, t, \varepsilon), \\ a_3^\varepsilon(x, t) &= a_3(x, t, \varepsilon), & f^\varepsilon(x, t) &= f(x, t, \varepsilon), & r_0^\varepsilon(t) &= r_0(t, \varepsilon), & r_1^\varepsilon(t) &= r_1(t, \varepsilon) \end{aligned} \quad (34)$$

of the coefficients $a(x, t)$, $a_1(x, t)$, $a_2(x, t)$, $a_3(x, t)$, $f(x, t)$, $r_0(t)$, and $r_1(t)$, respectively. Below we will also keep the notation

$$\begin{aligned} a(x, t) &= a(x, t, 0), & a_1(x, t) &= a_1(x, t, 0), & a_2(x, t) &= a_2(x, t, 0), \\ a_3(x, t) &= a_3(x, t, 0), & f(x, t) &= f(x, t, 0), & r_0(t) &= r_0(t, 0), & r_1(t) &= r_1(t, 0) \end{aligned} \quad (35)$$

for the non-perturbed coefficients. Similar notation will be used for the solutions

$$w^\varepsilon(x, t) = w(x, t, \varepsilon), \quad u^\varepsilon(x, t) = u(x, t, \varepsilon) \quad (36)$$

of the corresponding perturbed problems.

We are prepared to write down a perturbed problem to (1)–(3):

$$\partial_t^2 w^\varepsilon - a^\varepsilon(x, t)^2 \partial_x^2 w^\varepsilon + a_1^\varepsilon(x, t) \partial_t w^\varepsilon + a_2^\varepsilon(x, t) \partial_x w^\varepsilon + a_3^\varepsilon(x, t) w^\varepsilon = f^\varepsilon(x, t), \quad (37)$$

$$\begin{aligned} w^\varepsilon(x, t) &= w^\varepsilon(x, t + T), \\ \partial_t w^\varepsilon(x, t) &= \partial_t w^\varepsilon(x, t + T), \end{aligned} \quad (38)$$

$$\begin{aligned} \partial_x w^\varepsilon(0, t) &= r_0^\varepsilon(t) w^\varepsilon(0, t), \\ \partial_x w^\varepsilon(1, t) &= r_1^\varepsilon(t) w^\varepsilon(1, t). \end{aligned} \quad (39)$$

and the corresponding perturbed problem to (16)–(18):

$$\begin{aligned} \partial_t u_1^\varepsilon - a^\varepsilon(x, t) \partial_x u_1^\varepsilon + b_{11}^\varepsilon(x, t) u_1^\varepsilon + b_{12}^\varepsilon(x, t) u_2^\varepsilon &= f^\varepsilon(x, t) - [a_3^\varepsilon F^\varepsilon u^\varepsilon](x, t), \\ \partial_t u_2^\varepsilon + a^\varepsilon(x, t) \partial_x u_2^\varepsilon + b_{21}^\varepsilon(x, t) u_1^\varepsilon + b_{22}^\varepsilon(x, t) u_2^\varepsilon &= f^\varepsilon(x, t) - [a_3^\varepsilon F^\varepsilon u^\varepsilon](x, t), \end{aligned} \quad (40)$$

$$u_j^\varepsilon(x, t) = u_j^\varepsilon(x, t + T), \quad j = 1, 2, \quad (41)$$

$$\begin{aligned} u_1^\varepsilon(0, t) &= u_2^\varepsilon(0, t) + 2a^\varepsilon(0, t) r_0^\varepsilon(t) [G^\varepsilon u^\varepsilon](t), \\ u_2^\varepsilon(1, t) &= u_1^\varepsilon(1, t) - 2a^\varepsilon(1, t) r_1^\varepsilon(t) [F^\varepsilon u^\varepsilon](1, t), \end{aligned} \quad (42)$$

where the functions b_{ij}^ε and the operators F^ε and G^ε are given by (14) and (9)–(13) with a , a_1 , a_2 , a_3 , r_0 , and r_1 replaced by a^ε , a_1^ε , a_2^ε , a_3^ε , r_0^ε , and r_1^ε , respectively.

Theorem 1.5 *Assume (4) and (5). Let $\dim \mathcal{K}_w = 0$.*

(i) *Given a non-negative integer k , suppose*

$$\begin{aligned} a^\varepsilon &\in C^{k+1}([0, 1]; C_T^{k+2}), & a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon, f^\varepsilon &\in C^{k+1}([0, 1]; C_T^{k+1}), \\ r_0^\varepsilon, r_1^\varepsilon &\in C^{k+1}([0, 1]; C_T^k(\mathbb{R})) \end{aligned} \quad (43)$$

and assume that one of the conditions (30), (31), (32), and (33) is fulfilled. Then there exists $\varepsilon_0 \leq 1$ such that for all $\varepsilon \leq \varepsilon_0$ there exists a unique weak solution w^ε to (37)–(39). Moreover, it holds $w^\varepsilon \in C_T^{k+1}$, and the map $\varepsilon \in [0, \varepsilon_0) \mapsto w^\varepsilon \in C^{k-\gamma}$ is C^γ -smooth for any non-negative integer $\gamma \leq k$.

(ii) *Assume that a^ε is t -independent and $a^\varepsilon, a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon, r_0^\varepsilon, r_1^\varepsilon, f^\varepsilon$ are C^∞ -smooth. Suppose one of the conditions (26) and (27). Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ there exists a unique weak solution w^ε to (37)–(39). Moreover, for all $k \in \mathbb{N}$ it holds $w^\varepsilon \in C_T^k$, and the map $\varepsilon \in [0, \varepsilon_0) \mapsto w^\varepsilon \in C_T^k$ is C^∞ -smooth.*

Remark 1.6 Theorem 1.4 claims that, under a number of conditions ruling out resonances, more regular data ensure more regular solutions. This entails, in particular, that under the conditions of Theorem 1.5 the kernel of the operator of the problem (1)–(3) in C_T^{k+1} equals \mathcal{K}_w . This makes the assumption $\dim \mathcal{K}_w = 0$ of Theorem 1.5 rather natural.

Remark 1.7 For the sake of brevity in Theorem 1.5 we did not consider the special case when the coefficient a is ε -independent. In this case there is no loss of smoothness, i.e. the dependence on ε of the partial derivatives of the solution w^ε is as smooth as the dependence on ε of w^ε itself. Furthermore, the smooth dependence of w^ε and its partial derivatives on ε can be easily obtained by applying the classical Implicit Function Theorem. Specifically, if (43) is satisfied and if a is ε -independent, then the map $\varepsilon \in [0, \varepsilon_0) \mapsto w^\varepsilon \in \mathcal{C}^{k+1}$ is C^{k+1} -smooth.

We hope that Theorems 1.2, 1.4, and 1.5 will make possible developing a theory of local smooth continuation and bifurcation of time-periodic solutions to general semilinear boundary value second-order hyperbolic problems of a single space variable. Another interesting direction could be a non-smooth analysis of the discussed problems in the framework of algebras of generalized functions (see, e.g., [13]).

1.2 Related work

The paper [3] addresses time-periodic solutions to the homogeneous wave equation $\partial_t^2 w = \partial_x^2 w$ for $x \in [0, 1]$ with inhomogeneous boundary conditions $\beta \partial_t w(0, t) - \gamma \partial_x w(0, t) = f_0(t)$ and $\delta \partial_t w(1, t) + \gamma \partial_x w(1, t) = f_1(t)$, where the right-hand sides f_0 and f_1 are C^1 -smooth and periodic. It is shown that the solution C^1 -smoothly depends on the coefficients β, γ , and δ with respect to the L^2 -norm (see also [2] for a similar result). Remark that the smooth dependence result for a linear problem in general does not imply such a result for the corresponding semilinear problem because the superposition operator generated by a C^1 -smooth function is C^1 -smooth from L^2 into L^2 if and only if it is affine.

In the papers [5, 6, 7, 8, 14, 15] the Fredholm or isomorphism properties of the linear telegraph equation with constant coefficients are used to get solvability results for the corresponding semilinear problems.

In [10, 12] we investigated time-periodic problems for the general (with coefficients depending on space and time) first-order hyperbolic systems with reflection boundary conditions. We suggested a functional-analytic approach that allowed us to prove the Fredholm alternative in the scale of Sobolev-type spaces of periodic functions (in the autonomous case [10]) as well as in the space of continuous functions (in the non-autonomous case [12]). In the former case [10] we applied the Fourier series expansion, as in [17]. In the latter case, like to the present paper, we used a weak formulation based on integration along characteristic curves. The Fredholm solvability result was essentially used in the autonomous case to prove a smooth dependence on parameters and on the data. The general non-autonomous situation is much more complicated (and we address it here). The reason is that higher solution regularity, that is strongly related to the smooth dependence [10], can be achieved only if additional non-resonance conditions are fulfilled [9,

Section 2.3]. The main difference between the problem (16)–(18) and the problem that was investigated in [10, 12] is this: now a number of integral terms contribute into the system (16) as well as into the boundary conditions (18). To handle these terms, we will use the smoothing property proved in [9].

In [11] we applied our results from [12] to prove a Hopf bifurcation theorem for semi-linear hyperbolic systems.

2 Equivalence of the problems (1)–(3) and (16)–(18)

Here we prove that the problems (1)–(3) and (16)–(18) are equivalent in the sense of the classical solvability as well as in the sense of the weak solvability.

First show that if w is a classical solution to (1)–(3), then $u = (u_1, u_2)$ given by (15) is a classical solution to (16)–(18). With this aim we outline the derivation of (19). We will use the equalities

$$\partial_t w = \frac{u_1 + u_2}{2}, \quad \partial_x w = \frac{u_1 - u_2}{2a} \quad (44)$$

resulting from (15). Integrating the second one in x and then the first one in t , one gets

$$w(x, t) = [Iu](t) + [Ju](x, t) + w(0, 0). \quad (45)$$

In order to show that

$$w(0, 0) = Nu, \quad (46)$$

we first integrate the first equality of (15) in t over $[0, T]$, put there $x = 0$, and use the time-periodicity and the first equation from the boundary conditions (2). Consequently, we have

$$\int_0^T \left(u_1(0, t) - a(0, t)r_0(t)w(0, t) \right) dt = 0. \quad (47)$$

Then, calculating $w(0, t)$ by means of (45), the last equality can be expressed in the form

$$w(0, 0) \int_0^T a(0, t)r_0(t) dt = \int_0^T \frac{u_1(0, t) - u_2(0, t)}{2} dt - \int_0^T a(0, t)r_0(t)[Iu](t) dt,$$

which in the notation of (8) and (11) gives (46) as desired.

Now, on the account of (15) and (19), we easily come from (1)–(3) to (16)–(18) where the latter is satisfied in the classical sense.

Further our aim is to prove that w given by (19) belongs to C_T^2 whenever $u \in \mathcal{C}^1$ is a classical solution to (16)–(18). It suffices to show that $Ju \in C_T^2$ for every such u . By definition (10), we are done if we show that

$$\int_0^x \frac{u_j(\xi, t)}{a(\xi, t)} d\xi \in C_T^2 \quad (48)$$

for $j = 1, 2$. Let us do this for $j = 1$ (for $j = 2$ we apply a similar argument). Fix an arbitrary $u \in \mathcal{C}^1$ satisfying (16)–(18). Plugging the representation (23) for u_1 into the

integral (48), we see that we have to treat integral operators of two kinds, namely

$$[S_1 u_1](x, t) = \int_0^x \frac{c_1(0, \xi, t) u_2(0, \tau_1(0, \xi, t))}{a(\xi, t)} d\xi \quad (49)$$

and

$$[S_2 u_1](x, t) = \int_0^x \frac{1}{a(\xi, t)} \int_0^\xi d_1(0, \eta, t) b_{12}(\eta, \tau_1(\eta, \xi, t)) u_2(\eta, \tau_1(\eta, \xi, t)) d\eta d\xi \quad (50)$$

showing that they are smoothing and map C_T^1 into C_T^2 .

Denote by $\tau \in \mathbb{R} \mapsto \tilde{\tau}_i(\tau, x, t) \in [0, 1]$ the inverse of the equation of the i -th characteristic curve of (16) passing through the point $(x, t) \in [0, 1] \times \mathbb{R}$. Moreover, in the calculations below we will use the formulas:

$$\partial_t \tilde{\tau}_j(\tau, x, t) = (-1)^{j+1} a(x, t) \exp \int_t^\tau (-1)^j \partial_1 a(\tilde{\tau}_j(\eta, x, t), \eta) d\eta, \quad (51)$$

$$\partial_x \tilde{\tau}_j(\tau, x, t) = \exp \int_t^\tau (-1)^j \partial_1 a(\tilde{\tau}_j(\eta, x, t), \eta) d\eta. \quad (52)$$

By simple change of variables in (49), we get the following representation for $[S_1 u_1](x, t)$:

$$[S_1 u_1](x, t) = \int_t^{\tau_1(0, x, t)} \frac{c_1(0, \tilde{\tau}_1(t, 0, \tau), t) \partial_\tau \tilde{\tau}_1(t, 0, \tau) u_2(0, \tau)}{a(\tilde{\tau}_1(t, 0, \tau), t)} d\tau.$$

Taking into account (51) and smoothness assumptions on the initial data, we conclude that the right-hand side is a C^2 -function as desired.

It remains to treat (50). To this end, let

$$d_{12}(\eta, \xi, t) = a(\xi, t)^{-1} d_1(0, \eta, t) b_{12}(\eta, \tau_1(\eta, \xi, t)).$$

By Fubini's theorem,

$$[S_2 u_1](x, t) = \int_0^x \int_\eta^x d_{12}(\eta, \xi, t) u_2(\eta, \tau_1(\eta, \xi, t)) d\xi d\eta. \quad (53)$$

Hence,

$$\begin{aligned} \partial_t [S_2 u_1](x, t) &= \int_0^x \int_\xi^x \partial_t d_{12}(\eta, \xi, t) u_2(\eta, \tau_1(\eta, \xi, t)) d\eta d\xi \\ &\quad + \int_0^x \int_\eta^x d_{12}(\eta, \xi, t) \partial_t u_2(\eta, \tau_1(\eta, \xi, t)) d\xi d\eta. \end{aligned} \quad (54)$$

The first summand meets the C^1 -regularity. Let us show that this is the case for the second summand. On the account of the simple transformation

$$\partial_\xi u_2(\eta, \tau_1(\eta, \xi, t)) = \partial_2 u_2(\eta, \tau_1(\eta, \xi, t)) \partial_\xi \tau_1(\eta, \xi, t),$$

where $\partial_k g$ here and below denotes the derivative of g with respect to the k -th argument, we have

$$\begin{aligned}\partial_t u_2(\eta, \tau_1(\eta, \xi, t)) &= \partial_2 u_2(\eta, \tau_1(\eta, \xi, t)) \partial_t \tau_1(\eta, \xi, t) \\ &= \frac{\partial_t \tau_1(\eta, \xi, t)}{\partial_\xi \tau_1(\eta, \xi, t)} \partial_\xi u_2(\eta, \tau_1(\eta, \xi, t)).\end{aligned}\quad (55)$$

Here

$$\partial_x \tau_j(\xi, x, t) = \frac{(-1)^{j+1}}{a(x, t)} \exp \int_\xi^x (-1)^j \left(\frac{\partial_t a}{a^2} \right) (\eta, \tau_j(\eta, x, t)) d\eta, \quad (56)$$

$$\partial_t \tau_j(\xi; x, t) = \exp \int_\xi^x (-1)^j \left(\frac{\partial_t a}{a^2} \right) (\eta, \tau_j(\eta, x, t)) d\eta. \quad (57)$$

Then in the notation

$$\tilde{d}_{12}(\eta, \xi, t) = d_{12}(\eta, \xi, t) \frac{\partial_t \tau_1(\eta, \xi, t)}{\partial_\xi \tau_1(\eta, \xi, t)}$$

the second summand in (55) equals

$$\begin{aligned}& \int_0^x \int_\eta^x \tilde{d}_{12}(\eta, \xi, t) \partial_\xi u_2(\eta, \tau_1(\eta, \xi, t)) d\xi d\eta \\ &= \int_0^x \left[\tilde{d}_{12}(\eta, \xi, t) u_2(\eta, \tau_1(\eta, \xi, t)) \right]_{\xi=\eta}^x d\eta \\ &\quad - \int_0^x \int_\eta^x \partial_\xi \tilde{d}_{12}(\eta, \xi, t) u_2(\eta, \tau_1(\eta, \xi, t)) d\xi d\eta.\end{aligned}\quad (58)$$

We are prepared to conclude that the function $\partial_t [S_2 u_1](x, t)$ is continuously differentiable. Hence, $[S_2 u_1](x, t)$ has C^2 -regularity in t . To prove that it has C^2 -regularity also in x , we follow a similar argument, but this time we differentiate (53) in x .

The fact that w given by (19) satisfies (1)–(3) easily follows from (16)–(18).

The same argument works also to show the equivalence of the problems (1)–(3) and (16)–(18) in the sense of the weak solvability, the only difference being in applying the calculations performed by (53), (54), (55), and (58) with u_2 replaced by an arbitrary fixed sequence u_2^l tending to u_2 in C_T as $l \rightarrow \infty$. Passing to the limit as $l \rightarrow \infty$ in thus obtained analog of (58) finishes the proof.

3 Fredholm alternative: proof of Theorem 1.2

On the account of Section 2, we are done if we prove the Fredholm alternative for (16)–(18): First, $\dim \mathcal{K}_u < \infty$, where \mathcal{K}_u is the vector space of all weak solutions to (16)–(18) with $f = 0$. Second, the space of all $f \in C_T$ such that there exists a weak solution to (16)–(18) is a closed subspace of codimension $\dim \mathcal{K}_u$ in \mathcal{C} . Third, either $\dim \mathcal{K}_u > 0$ or for any $f \in C_T$ there exists exactly one weak solution u to (16)–(18).

To simplify further notation, in parallel with the notation $\tau_j(\xi, x, t)$ we will use its shortened form $\tau_j(\xi)$. The system (23)–(24) can be written as the operator equation

$$u = Bu + Au + Du + Rf, \quad (59)$$

where the linear bounded operators $B, A, D : \mathcal{C} \rightarrow \mathcal{C}$ and $R : C \rightarrow \mathcal{C}$ are defined by

$$\begin{aligned} [Bu](x, t) &= \left(c_1(0, x, t)u_2(0, \tau_1(0)), c_2(1, x, t)u_1(1, \tau_2(1)) \right) \\ [Au](x, t) &= \left(2c_1(0, x, t)a(0, \tau_1(0))r_0(\tau_1(0))[Gu](\tau_1(0)), \right. \\ &\quad \left. -2c_2(1, x, t)a(1, \tau_2(1))r_1(\tau_2(1))[Fu](1, \tau_2(1)) \right), \\ [Du](x, t) &= \left(-\int_0^x d_1(\xi, x, t)(b_{12}u_2 - [a_3Fu])(\xi, \tau_1(\xi))d\xi, \right. \\ &\quad \left. -\int_1^x d_2(\xi, x, t)(b_{21}u_1 - [a_3Fu])(\xi, \tau_2(\xi))d\xi \right), \\ [Rf](x, t) &= \left(\int_0^x d_1(\xi, x, t)f(\xi, \tau_1(\xi))d\xi, \int_1^x d_2(\xi, x, t)f(\xi, \tau_2(\xi))d\xi \right). \end{aligned}$$

We have to show that the operator $I - B - A - D$ is Fredholm of index zero from \mathcal{C} to \mathcal{C} . First we prove the bijectivity of $I - B$:

Lemma 3.1 *If one of the conditions (26) and (27) is fulfilled, then $I - B$ is bijective from \mathcal{C} to \mathcal{C} .*

Proof. Suppose (26). Let $g = (g_1, g_2) \in \mathcal{C}$ be arbitrary given. We have $u = Bu + g$ or, the same,

$$u_1(x, t) = c_1(0, x, t)u_2(0, \tau_1(0)) + g_1(x, t), \quad u_2(x, t) = c_2(1, x, t)u_1(1, \tau_2(1)) + g_2(x, t) \quad (60)$$

if and only if

$$\begin{aligned} u_1(x, t) &= c_1(0, x, t)[c_2(1, 0, \tau_1(0))u_1(1, \tau_2(1, 0, \tau_1(0))) + g_2(0, \tau_1(0))] + g_1(x, t), \\ u_2(x, t) &= c_2(1, x, t)u_1(1, \tau_2(1)) + g_2(x, t). \end{aligned} \quad (61)$$

Observe that it suffices to show the existence of a unique continuous solution $t \in [0, T] \rightarrow u_1(1, t) \in \mathbb{R}$. Putting $x = 1$ in the first equation of (61), we get

$$u_1(1, t) = c_1(0, 1, t)c_2(1, 0, \tau_1(0, 1, t))u_1(1, \tau_2(1, 0, \tau_1(0, 1, t))) + \tilde{g}(1, t), \quad (62)$$

where $\tilde{g}(x, t) = c_1(0, x, t)g_2(0, \tau_1(0)) + g_1(x, t)$. Putting then $t = \tau_1(1, 0, \tau_2(0, 1, \tau))$, we come to another writing of (62), namely

$$\begin{aligned} &[c_1(0, 1, \tau_1(1, 0, \tau_2(0, 1, \tau)))c_2(1, 0, \tau_2(0, 1, \tau))]u_1(1, \tau) \\ &= (u_1 - \tilde{g})(1, \tau_1(1, 0, \tau_2(0, 1, \tau))). \end{aligned} \quad (63)$$

Here we used the identity $\tau_2(1, 0, \tau_1(0, 1, \tau_1(1, 0, \tau_2(0, 1, \tau)))) \equiv \tau$, being true for all $\tau \in \mathbb{R}$. Due to the Banach fixed point argument, Equations (62) and (63) are uniquely solvable under the contraction conditions, respectively,

$$c_1(0, 1, t)c_2(1, 0, \tau_1(0, 1, t)) < 1 \text{ for all } t \in [0, T]$$

and

$$[c_1(0, 1, \tau_1(1, 0, \tau_2(0, 1, \tau)))c_2(1, 0, \tau_2(0, 1, \tau))]^{-1} < 1 \text{ for all } \tau \in [0, T].$$

Since the latter is equivalent to $[c_1(0, 1, t)c_2(1, 0, \tau_1(0, 1, t))]^{-1} < 1$ for all $t \in [0, T]$, we immediately meet assumption (26). The proof under the assumption (26) is thereby complete.

The proof under the assumption (27) follows along the same line as above, the only difference being in using instead of (61) another equivalent form of (60), namely

$$\begin{aligned} u_1(x, t) &= c_1(0, x, t)u_2(0, \tau_1(0)) + g_1(x, t), \\ u_2(x, t) &= c_2(1, x, t)[c_1(0, 1, \tau_2(1))u_2(0, \tau_1(0, 1, \tau_2(1))) + g_1(1, \tau_2(1))] + g_2(x, t), \end{aligned} \quad (64)$$

and putting $x = 0$ in the latter. \square

Returning to the operator $I - B - A - D$, we would like to emphasize that the operators A and D are not compact from \mathcal{C} to \mathcal{C} , in general, because they are partial integral operators (other kinds of partial integral operators are investigated in [1]). By Lemma 3.1, the operator $I - B - A - D$ is Fredholm of index zero from \mathcal{C} to \mathcal{C} if and only if $I - (I - B)^{-1}(A + D)$ is Fredholm of index zero from \mathcal{C} to \mathcal{C} . Then, on the account of Fredholmness criterion [4, Theorem XIII.5.2], we are done if we prove the following statement:

Lemma 3.2 *The operator $[(I - B)^{-1}(A + D)]^2$ is compact from \mathcal{C} to \mathcal{C} .*

Proof. Due to the boundedness of the operator $(I - B)^{-1}$, it is sufficient to prove that

$$(A + D)(I - B)^{-1}(A + D) \text{ is compact from } \mathcal{C} \text{ to } \mathcal{C}. \quad (65)$$

Since

$$(A + D)(I - B)^{-1}(A + D) = (A + D)^2 + (A + D)B(I - B)^{-1}(A + D),$$

the statement (65) will be proved if we show that

$$(A + D)^2 \text{ and } (A + D)B \text{ are compact from } \mathcal{C} \text{ to } \mathcal{C}. \quad (66)$$

By the Arzela-Ascoli theorem, \mathcal{C}^1 is compactly embedded into \mathcal{C} . Hence, for (66) it suffices to show that

$$(A + D)^2 \text{ and } (A + D)B \text{ map continuously } \mathcal{C} \text{ into } \mathcal{C}^1. \quad (67)$$

To reach (67), we will prove the following smoothing property:

$$A^2, D^2, AD, DA, AB, \text{ and } DB \text{ map continuously } \mathcal{C} \text{ into } \mathcal{C}^1. \quad (68)$$

Let us start with the operator A^2 . Using the definition (60), we are done if we show that

$$GA \text{ and } FA \text{ map continuously } \mathcal{C} \text{ into } \mathcal{C}^1. \quad (69)$$

On the account of the definition (12) of G and the continuous embedding of $C_T^1(\mathbb{R})$ into \mathcal{C}^1 , the operator G and, hence, the operator GA maps continuously \mathcal{C} into \mathcal{C}^1 . Moreover, by the definition (13) of F , to get (69) for FA we only need to handle the operator

$$[J Au](1, t) = \int_0^1 \frac{[Au]_1(x, t) - [Au]_2(x, t)}{2a(x, t)} dx.$$

Again, by the definition of A , we are left with the integral

$$\int_0^1 \frac{[Au]_2(x, t)}{2a(x, t)} dx$$

or, even more, with the integral

$$\begin{aligned} & \int_0^1 \frac{c_2(1, x, t)a(1, \tau_2(1))r_1(\tau_2(1))}{2a(x, t)} [Ju](1, \tau_2(1)) dx \\ &= \int_0^1 \frac{c_2(1, x, t)a(1, \tau_2(1))r_1(\tau_2(1))}{2a(x, t)} \int_0^1 \left(\frac{u_1 - u_2}{2a} \right) (\xi, \tau_2(1)) d\xi dx \\ &= \int_0^1 \int_{\tau_2(1, 0, t)}^t \frac{c_2(1, \tilde{\tau}_2(t, 1, \tau), t)a(1, \tau)r_1(\tau)\partial_\tau \tilde{\tau}_2(t, 1, \tau)}{2a(\tilde{\tau}_2(t, 1, \tau), t)} \left(\frac{u_1 - u_2}{2a} \right) (\xi, \tau) d\tau d\xi, \end{aligned}$$

where $\partial_\tau \tilde{\tau}_2(t, 1, \tau)$ is given by (52). The right-hand side of the latter equality has the desired smoothing property, what finishes the proof of (69).

Next we prove (68) for D^2 . Taking into account the density of \mathcal{C}^1 in \mathcal{C} , we are done if we show that there is a constant $C > 0$ such that

$$\|\partial_x D^2 u\|_\infty + \|\partial_t D^2 u\|_\infty \leq C \|u\|_\infty \quad (70)$$

for all $u \in \mathcal{C}^1$. Using the definitions of D and F and the smoothing property of G mentioned above, we only need to treat integral operators of two types contributing into D^2 . Thus, the integral operator of the first type

$$\begin{aligned} & \int_0^x d_1(\xi, x, t) a_3(\xi, \tau_1(\xi)) \int_0^\xi \left(\frac{u_1 - u_2}{2a} \right) (\eta, \tau_1(\xi)) d\eta d\xi \\ &= - \int_0^x \int_{\tau_1(\eta)}^t d_1(\tilde{\tau}_1(\tau), x, t) a_3(\tilde{\tau}_1(\tau), \tau) a(\tilde{\tau}_1(\tau), \tau) \left(\frac{u_1 - u_2}{2a} \right) (\eta, \tau) d\tau d\eta, \end{aligned} \quad (71)$$

where $\tilde{\tau}_1(\tau) = \tilde{\tau}_1(\tau, x, t)$, maps continuously \mathcal{C} into \mathcal{C}^1 , what immediately entails the estimate of kind (70) for it. It remains to prove the upper bound $C \|u\|_\infty$ for the integral of the type

$$\begin{aligned} & \int_0^x \int_0^\xi d_{12}(\xi, \eta, x, t) u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\eta d\xi \\ &= \int_x^1 \int_0^\eta d_{12}(\xi, \eta, x, t) u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\xi d\eta \\ &+ \int_x^1 \int_0^x d_{12}(\xi, \eta, x, t) u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\xi d\eta \end{aligned} \quad (72)$$

with

$$d_{12}(\xi, \eta, x, t) = d_1(\xi, x, t)d_2(\eta, \xi, \tau_1(\xi))b_{12}(\xi, \tau_1(\xi))b_{21}(\eta, \tau_2(\eta; \xi, \tau_1(\xi))).$$

Note that

$$\begin{aligned} & (\partial_t - a(x, t)\partial_x) \int_0^x \int_1^\xi d_{12}(\xi, \eta, x, t)u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\eta d\xi \\ &= -a(x, t) \int_1^x d_{12}(x, \eta, x, t)u_1(\eta, \tau_2(\eta)) d\eta, \end{aligned}$$

where the derivatives are considered in a distributional sense. Hence, to derive (70) with $[D^2u](x, t)$ replaced by (72), it is sufficient to prove the estimate $\|\partial_t D^2u\|_\infty \leq C\|u\|_\infty$ satisfying uniformly in $u \in \mathcal{C}^1$. Thus, we differentiate (72) with respect to t (without loss of generality we illustrate our argument only on the first summand in the right-hand side of (72)) and get

$$\begin{aligned} & \int_0^x \int_0^\eta \partial_t d_{12}(\xi, \eta, x, t)u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\xi d\eta \\ &+ \int_0^x \int_0^\eta d_{12}(\xi, \eta, x, t)\partial_t u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\xi d\eta. \end{aligned} \quad (73)$$

The first summand obviously fits the desired estimate. To handle the second one, we compute

$$\begin{aligned} & \partial_\xi u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) \\ &= \partial_2 u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) [\partial_2 \tau_2(\eta; \xi, \tau_1(\xi)) + \partial_3 \tau_2(\eta; \xi, \tau_1(\xi))\partial_\xi \tau_1(\xi)]. \end{aligned}$$

Hence, applying (20), (56), and (57) gives

$$\begin{aligned} & \partial_t u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) = \partial_2 u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi)))\partial_3 \tau_2(\eta; \xi, \tau_1(\xi))\partial_t \tau_1(\xi) \\ &= \frac{\partial_\xi u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi)))\partial_3 \tau_2(\eta; \xi, \tau_1(\xi))\partial_t \tau_1(\xi)}{\partial_2 \tau_2(\eta; \xi, \tau_1(\xi)) + \partial_3 \tau_2(\eta; \xi, \tau_1(\xi))\partial_\xi \tau_1(\xi)} \\ &= -\frac{1}{2}a(\xi, \tau_1(\xi))\partial_t \tau_1(\xi)\partial_\xi u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))). \end{aligned} \quad (74)$$

Then, using the notation

$$\tilde{d}_{12}(\xi, \eta, x, t) = -\frac{1}{2}d_{12}(\xi, \eta, x, t)a(\xi, \tau_1(\xi))\partial_t \tau_1(\xi),$$

the second summand in (73) equals

$$\begin{aligned} & \int_0^x \int_0^\eta \tilde{d}_{12}(\xi, \eta, x, t)\partial_\xi u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\xi d\eta \\ &= \int_0^x \left[\tilde{d}_{12}(\xi, \eta, x, t)u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) \right]_{\xi=0}^{\xi=\eta} d\eta \\ &- \int_0^x \int_0^\eta \partial_\xi \tilde{d}_{12}(\xi, \eta, x, t)u_1(\eta, \tau_2(\eta; \xi, \tau_1(\xi))) d\xi d\eta, \end{aligned}$$

what immediately entails the desired estimate. We therefore finished with the estimate (70).

Further we prove (68) for the operator AD . As above, due to the definition of A , we are reduced to give the proof for the operator FD only. On the account of the definition of F , the latter will be proved once we handle the operator JD . Thus,

$$\begin{aligned} [JD u](x, t) = & - \int_0^x \frac{1}{2a(\xi, t)} \int_0^\xi d_1(\eta, \xi, t) \left(b_{12}u_2 - [a_3 F u] \right)(\eta, \tau_1(\eta, \xi, t)) d\eta d\xi \\ & + \int_0^x \frac{1}{2a(\xi, t)} \int_1^\xi d_2(\eta, \xi, t) \left(b_{21}u_1 - [a_3 F u] \right)(\eta, \tau_2(\eta, \xi, t)) d\eta d\xi. \end{aligned}$$

After changing the order of integration and making a simple change of variables the first summand in the right-hand side (and similarly for the second summand) can be written in the form

$$- \int_0^x \int_t^{\tau_1(\eta)} \frac{1}{2a(\tilde{\tau}_1(t, \eta, \tau), t)} d_1(\eta, \tilde{\tau}_1(t, \eta, \tau), t) \left(b_{12}u_2 - [a_3 F u] \right)(\eta, \tau) d\tau d\eta$$

allowing to state the desired smoothing property.

On the next step we treat the operator DA . For instance, for $[DAu]_1$ (and similarly for $[DAu]_2$) we have

$$\begin{aligned} [DAu]_1(x, t) = & \int_0^x d_1(\xi, x, t) \left(2b_{12}(\xi, \tau_1(\xi))c_2(1, \xi, \tau_1(\xi))a(1, \tau_2(1, \xi, \tau_1(\xi))) \right. \\ & \left. \times r_1(\tau_2(1, \xi, \tau_1(\xi))) [Fu](1, \tau_2(1, \xi, \tau_1(\xi))) - [a_3 F Au] \right)(\xi, \tau_1(\xi)) d\xi. \end{aligned}$$

Again, by the definition of F , we are done if we prove the smoothing property (70) for the latter expression but with J in place of F . Here one can apply the same argument as in (71) (changing the order of integration and using the changing of variables $\tau = \tau_1(\xi)$).

Turning back to (68), we further proceed with the operator AB . By the definition of A, B , and F , it suffices to show that the operator JB maps continuously \mathcal{C} into C_T^1 . Indeed,

$$\begin{aligned} [JBu](x, t) = & \int_0^x \frac{c_1(0, \xi, t)u_2(0, \tau_1(0, \xi, t)) - c_2(1, \xi, t)u_1(1, \tau_2(1, \xi, t))}{2a(\xi, t)} d\xi \\ = & \int_t^{\tau_1(0)} \frac{c_1(0, \tilde{\tau}_1(t, 0, \tau), t)u_2(0, \tau)}{2a(\tilde{\tau}_1(t, 0, \tau), t)} \partial_\tau \tilde{\tau}_1(t, 0, \tau) d\tau \\ & - \int_{\tau_2(1, 0, t)}^{\tau_2(1)} \frac{c_2(1, \tilde{\tau}_2(t, 1, \tau), t)u_1(1, \tau)}{2a(\tilde{\tau}_2(t, 1, \tau), t)} \partial_\tau \tilde{\tau}_2(t, 1, \tau) d\tau. \end{aligned} \tag{75}$$

The desired property for AB now easily follows from the smoothness assumptions (25) and (51).

Finally, we prove (68) for the operator DB . Denote by $\tilde{x}(\tau, x, t)$ the value of ξ where the characteristics $\tau_2(\xi, 1, \tau)$ and $\tau_1(\xi, x, t)$ intersect, namely

$$\tau_2(\tilde{x}(\tau, x, t), 1, \tau) = \tau_1(\tilde{x}(\tau, x, t), x, t).$$

It follows from (25) that the function $\tilde{x}(\tau, x, t)$ is continuously differentiable in its arguments. Furthermore,

$$\begin{aligned}\partial_\tau \tilde{x}(\tau, x, t) &= \frac{\partial_3 \tau_2(\tilde{x}(\tau, x, t), 1, \tau)}{\partial_1 \tau_1(\tilde{x}(\tau, x, t), x, t) - \partial_1 \tau_2(\tilde{x}(\tau, x, t), 1, \tau)} \\ &= -\frac{a(\tilde{x}(\tau, x, t), \tau_1(\tilde{x}(\tau, x, t)))}{2} \exp \int_{\tilde{x}(\tau, x, t)}^1 \left(\frac{\partial_t a}{a^2} \right) (\eta, \tau_2(\eta; 1, \tau)) d\eta.\end{aligned}\quad (76)$$

Here we used (20) and (56). Similarly,

$$\partial_x \tilde{x}(\tau, x, t) = \frac{a(\tilde{x}(\tau, x, t), \tau_1(\tilde{x}(\tau, x, t)))}{2a(x, t)} \exp \int_x^{\tilde{x}(\tau, x, t)} \left(\frac{\partial_t a}{a^2} \right) (\eta, \tau_1(\eta)) d\eta, \quad (77)$$

$$\partial_t \tilde{x}(\tau, x, t) = \frac{a(\tilde{x}(\tau, x, t), \tau_1(\tilde{x}(\tau, x, t)))}{2} \exp \int_x^{\tilde{x}(\tau, x, t)} \left(\frac{\partial_t a}{a^2} \right) (\eta, \tau_1(\eta)) d\eta. \quad (78)$$

By the definitions of D and B as well as (71), to handle DB , it remains to treat the integrals of the type

$$\begin{aligned}& \int_0^x d_1(\xi, x, t) b_{12}(\xi, \tau_1(\xi)) c_2(1, \xi, \tau_1(\xi)) u_1(1, \tau_2(1, \xi, \tau_1(\xi))) d\xi \\ &= \int_{\tau_2(1, 0, \tau_1(0))}^{\tau_2(1)} d_1(\tilde{x}(\tau, x, t), x, t) b_{12}(\tilde{x}(\tau, x, t), \tau_1(\tilde{x}(\tau, x, t))) \\ &\quad \times c_2(1, \tilde{x}(\tau, x, t), \tau_1(\tilde{x}(\tau, x, t))) u_1(1, \tau) \partial_\tau \tilde{x}(\tau, x, t) d\tau.\end{aligned}$$

On the account of (76), (77), and (78), we immediately come to the desired conclusion, what completes the proof of the lemma. \square

4 Higher regularity of solutions: proof of Theorem 1.4

Here we address the issue of a higher regularity of weak solutions in the case of a higher regularity of the coefficients in (1) and (3) and an additional number of non-resonance conditions. We therefore let (4), (5), (29), and one of the conditions (30), (31), (32), and (33) to be fulfilled.

First note that the statement (ii) of Theorem 1.4 is a straightforward consequence of the statement (i), since in the case of stationary a we have $c_j^l \equiv c_j$ for all $l \geq 1$ and, hence, condition (26) implies either (30) or (31) for any positive integer k .

On the account of the equivalence of the problems (1)–(3) and (16)–(18) stated in Section 2, we are reduced to prove that any weak solution u to (16)–(18) reaches C^k -regularity. To this end, we introduce a couple of Banach spaces: Given a positive integer i , set

$$\tilde{C}_T^i = \{u \in C_T : \partial_t^i u \in C_T\} \quad \text{and} \quad \tilde{\mathcal{C}}^i = \tilde{C}_T^i \times \tilde{C}_T^i.$$

Let $u \in \mathcal{C}$ be an arbitrary fixed weak solution to (16)–(18). The proof is by induction on the order of regularity (the order of continuous differentiability) of the solutions.

Base case: $u \in \mathcal{C}^1$. First show that the generalized directional derivatives $(\partial_t - a\partial_x)u_1$ and $(\partial_t + a\partial_x)u_2$, where ∂_x and ∂_t denote the generalized derivatives, is a continuous function; this reduces our task to proving that $u \in \tilde{\mathcal{C}}^1$. Take an arbitrary sequence $u^l \in \mathcal{C}^1$ approaching u in \mathcal{C} and an arbitrary smooth function $\varphi : (0, 1) \times (0, T) \rightarrow \mathbb{R}$ with compact support. Then

$$\begin{aligned}
\langle (\partial_t - a\partial_x)u_1, \varphi \rangle &= \langle u_1, -\partial_t\varphi + \partial_x(a\varphi) \rangle = \lim_{l \rightarrow \infty} \langle u_1^l, -\partial_t\varphi + \partial_x(a\varphi) \rangle \\
&= \lim_{l \rightarrow \infty} \left\langle c_1(0, x, t) \left[u_2^l(0, \tau_1(0)) + 2a(0, \tau_1(0))r_0(\tau_1(0))[Gu^l](\tau_1(0)) \right] \right. \\
&\quad - \int_0^x d_1(\xi, x, t)b_{12}(\xi, \tau_1(\xi))u_2^l(\xi, \tau_1(\xi))d\xi \\
&\quad \left. + \int_0^x d_1(\xi, x, t) \left(f(\xi, \tau_1(\xi)) - [a_3Fu^l](\xi, \tau_1(\xi)) \right) d\xi, -\partial_t\varphi + \partial_x(a\varphi) \right\rangle \\
&= \lim_{l \rightarrow \infty} \langle -b_{11}(x, t)u_1^l - b_{12}(x, t)u_2^l + f(x, t) - [a_3Fu^l](x, t), \varphi \rangle \\
&= \langle -b_{11}(x, t)u_1 - b_{12}(x, t)u_2 + f(x, t) - [a_3Fu](x, t), \varphi \rangle
\end{aligned}$$

as desired. Here we used the formula

$$(\partial_t + (-1)^j a(x, t)\partial_x)\psi(\tau_j(\xi, x, t)) = 0$$

being true for all $j = 1, 2$, $\xi, x \in [0, 1]$, $t \in \mathbb{R}$, and any $\psi \in C^1(\mathbb{R})$. Similarly we compute the generalized directional derivative $(\partial_t + a\partial_x)u_2$.

Therefore, the beginning step of the induction will follow from the fact that $u \in \tilde{\mathcal{C}}^1$. To prove the latter, we substitute (59) into the second and the third summands of (59) and get

$$u = Bu + (A^2 + AB + AD + DB + DA + D^2)u + (I + A + D)Rf. \quad (79)$$

On the account of the smoothing property (68) of the operators A^2 , AB , AD , DB , DA , and D^2 , we are done if we show the bijectivity of $I - B \in \mathcal{L}(\mathcal{C})$ from $\tilde{\mathcal{C}}^1$ to $\tilde{\mathcal{C}}^1$. In other words, we have to show that the system (61) is uniquely solvable in $\tilde{\mathcal{C}}^1$ for any $(g_1, g_2) \in \tilde{\mathcal{C}}^1$. Following the argument as in the proof of Lemma 3.1, the latter is true iff

$$I - B' \text{ is bijective from } C_T^1(\mathbb{R}) \text{ to } C_T^1(\mathbb{R}), \quad (80)$$

where the operator $B' \in \mathcal{L}(C_T(\mathbb{R}))$ is given by

$$[B'v](t) = c_1(0, 1, t)c_2(1, 0, \tau_1(0, 1, t))v(\tau_2(1, 0, \tau_1(0, 1, t))). \quad (81)$$

Now we aim to show (80) whenever one of the conditions (30), (31), (32), and (33) with $k = 1$ is satisfied. We will prove the desired statement under the condition (30). A similar argument combined with the one used in the proof of Lemma 3.1 works in the case of (31), (32), or (33) as well.

Thus, following the idea of [16] (see also [12]) used to establish the solution regularity for first-order hyperbolic PDEs, given $\beta > 0$, we norm the space $C_T^1(\mathbb{R})$ with

$$\|v\|_{C_T^1(\mathbb{R})} = \|v\|_\infty + \beta \|\partial_t v\|_\infty. \quad (82)$$

Note that $C_T^1(\mathbb{R})$ endowed with (82) is a Banach space. Given $l = 0, 1, 2, \dots$, set

$$\begin{aligned} q_l &= \max_{x,y \in [0,1]} \max_{t \in \mathbb{R}} |c_1^l(0, 1, t) c_2^l(1, 0, \tau_1(0, 1, t))|, \\ q'_l &= \max_{x,y \in [0,1]} \max_{t \in \mathbb{R}} \left| \frac{d}{dt} [c_1^l(0, 1, t) c_2^l(1, 0, \tau_1(0, 1, t))] \right|. \end{aligned} \quad (83)$$

We are reduced to prove that there exist constants $\beta < 1$ and $\gamma < 1$ such that

$$\|B'v\|_\infty + \beta \left\| \frac{d}{dt} B'v \right\|_\infty \leq \gamma (\|v\|_\infty + \beta \|v'\|_\infty) \text{ for all } v \in C_T^1(\mathbb{R}).$$

Taking into account that $\|B'\|_{\mathcal{L}(C_T(\mathbb{R}))} \leq q_0 < 1$ (by assumption (26)), the latter estimate will be proved if we show that

$$\left\| \frac{d}{dt} B'v \right\|_\infty \leq \frac{\gamma - q_0}{\beta} \|v\|_\infty + \gamma \|v'\|_\infty \text{ for all } v \in C_T^1(\mathbb{R}). \quad (84)$$

Since

$$\begin{aligned} \frac{d}{dt} [B'v](t) &= \frac{d}{dt} [c_1(0, 1, t) c_2(1, 0, \tau_1(0, 1, t))] v(\tau_2(1, 0, \tau_1(0, 1, t))) \\ &\quad + c_1(0, 1, t) c_2(1, 0, \tau_1(0, 1, t)) \partial_2 \tau_2(1, 0, \tau_1(0, 1, t)) \partial_t \tau_1(0, 1, t) v'(\tau_2(1, 0, \tau_1(0, 1, t))), \end{aligned}$$

we get

$$\left\| \frac{d}{dt} B'v \right\|_\infty \leq q'_0 \|v\|_\infty + q_1 \|v'\|_\infty. \quad (85)$$

By the assumption (30) we have $q_0 < 1$ and $q_1 < 1$. Fix γ such that $\max\{q_0, q_1\} < \gamma < 1$. Then choose β so small that

$$q'_0 \leq \frac{\gamma - q_0}{\beta}.$$

Then (85) implies (84) as desired. The proof of the base case of the induction is therewith complete. Notice that the function u now satisfies the system (16) pointwise.

Induction assumption: $u \in \mathcal{C}^i$ for some $1 \leq i \leq k-1$.

Induction step: $u \in \mathcal{C}^{i+1}$. By the induction assumption, the function u satisfies the following system pointwise:

$$\begin{aligned} (\partial_t - a \partial_x) \partial_t^{i-1} u_1 &= - \left(b_{11} + (i-1) \frac{\partial_t a}{a} \right) \partial_t^{i-1} u_1 - b_{12} \partial_t^{i-1} u_2 \\ &\quad + f_{1,i-1}(x, t, u, \partial_t u, \dots, \partial_t^{i-2} u) - [P_{i-1}u](x, t) - [a_3 J \partial_t^{i-1} u](x, t), \\ (\partial_t + a \partial_x) \partial_t^{i-1} u_2 &= -b_{21} \partial_t^{i-1} u_1 - \left(b_{22} - (i-1) \frac{\partial_t a}{a} \right) \partial_t^{i-1} u_2 \\ &\quad + f_{2,i-1}(x, t, u, \partial_t u, \dots, \partial_t^{i-2} u) - [P_{i-1}u](x, t) - [a_3 J \partial_t^{i-1} u](x, t) \end{aligned} \quad (86)$$

with certain continuously differentiable functions $f_{1,i-1}$ and $f_{2,i-1}$ such that $f_{1,0} = f_{2,0} \equiv 0$ and with the operators $P_i \in \mathcal{L}(\mathcal{C}^i)$ defined by

$$\begin{aligned} [P_i u](x, t) &= \partial_t^i a_3 [Iu](t) + \frac{1}{2} \partial_t^{i-1} [a_3(u_1(0, t) + u_2(0, t))] \\ &\quad + \partial_t^{i-1} \int_0^x \partial_t \left(\frac{a_3(x, t)}{a(\xi, t)} \right) \frac{u_1(\xi, t) - u_2(\xi, t)}{2} d\xi \end{aligned} \quad (87)$$

such that $[P_0 u](x, t) \equiv 0$. Using (29) and the induction assumption, we see that the right-hand side of (86) is continuously differentiable in t . Hence, the left-hand side is continuously differentiable in t as well. Note that the latter does not imply the existence of the pointwise derivatives $\partial_t^{i+1} u_j$ and $\partial_x \partial_t^i u_j$ for $j = 1, 2$, but only the distributional ones. Set $v = \partial_t^i u$. Then the continuous function v satisfies the system

$$\begin{aligned} (\partial_t - a \partial_x) v_1 &= - \left(b_{11} + i \frac{\partial_t a}{a} \right) v_1 - b_{12} v_2 \\ &\quad + f_{1i}(x, t, u, \partial_t u, \dots, \partial_t^{i-1} u) - [P_i u](x, t) - [a_3 Jv](x, t), \\ (\partial_t + a \partial_x) v_2 &= -b_{21} v_1 - \left(b_{22} - i \frac{\partial_t a}{a} \right) v_2 \\ &\quad + f_{2i}(x, t, u, \partial_t u, \dots, \partial_t^{i-1} u) - [P_i u](x, t) - [a_3 Jv](x, t) \end{aligned} \quad (88)$$

in a distributional sense and the conditions

$$v_j(x, t) = v_j(x, t + T), \quad j = 1, 2, \quad (89)$$

and

$$\begin{aligned} v_1(0, t) &= v_2(0, t) + 2 \partial_t^i (a(0, t) r_0(t)) [Gu](t) + \partial_t^{i-1} [a(0, t) r_0(t) (u_1(0, t) + u_2(0, t))], \\ v_2(1, t) &= v_1(1, t) - 2 \partial_t^i (a(1, t) r_1(t)) [Gu](t) - \partial_t^{i-1} [a(1, t) r_1(t) (u_1(0, t) + u_2(0, t))] \\ &\quad - \partial_t^{i-1} \int_0^x \partial_t \left(\frac{a(1, t) r_1(t)}{a(\xi, t)} \right) (u_1(\xi, t) - u_2(\xi, t)) d\xi - 2a(1, t) r_1(t) [Jv](1, t) \end{aligned} \quad (90)$$

pointwise. We rewrite the system (88)–(90) in the following form:

$$(\partial_t - a \partial_x) v_1 = - \left(b_{11} + i \frac{\partial_t a}{a} \right) v_1 - b_{12} v_2 - [a_3 Jv](x, t) + [Q_i u]_1(x, t), \quad (91)$$

$$(\partial_t + a \partial_x) v_2 = -b_{21} v_1 - \left(b_{22} - i \frac{\partial_t a}{a} \right) v_2 - [a_3 Jv](x, t) + [Q_i u]_2(x, t),$$

$$v_j(x, t) = v_j(x, t + T), \quad j = 1, 2, \quad (92)$$

$$\begin{aligned} v_1(0, t) &= v_2(0, t) + [S_i u]_1(t), \\ v_2(1, t) &= v_1(1, t) + [S_i u]_2(t) - 2a(1, t) r_1(t) [Jv](1, t), \end{aligned} \quad (93)$$

where the operators $Q_i, S_i \in \mathcal{L}(\mathcal{C}^i)$ are defined by

$$\begin{aligned} [Q_i u](x, t) &= \left(f_{1i}(x, t, u, \partial_t u, \dots, \partial_t^{i-1} u) - [P_i u](x, t), \right. \\ &\quad \left. f_{2i}(x, t, u, \partial_t u, \dots, \partial_t^{i-1} u) - [P_i u](x, t) \right), \end{aligned} \quad (94)$$

$$\begin{aligned} [S_i u](t) &= \left(2\partial_t^i(a(0, t)r_0(t)) [Gu](t) + \partial_t^{i-1}(a(0, t)r_0(t)(u_1(0, t) + u_2(0, t))) \right. \\ &\quad - 2\partial_t^i(a(1, t)r_1(t)) [Gu](t) - \partial_t^{i-1}[a(1, t)r_1(t)(u_1(0, t) + u_2(0, t))] \\ &\quad \left. - \partial_t^{i-1} \int_0^x \partial_t \left(\frac{a(1, t)r_1(t)}{a(\xi, t)} \right) (u_1(\xi, t) - u_2(\xi, t)) d\xi \right). \end{aligned} \quad (95)$$

It follows, in particular, that

$$\begin{aligned} &\text{Given } u \in \mathcal{C}^i, \text{ the functions } [Q_i u]_1(x, t), [Q_i u]_2(x, t), \\ &[S_i u]_1(t), \text{ and } [S_i u]_2(t) \text{ are continuously differentiable.} \end{aligned} \quad (96)$$

We intend to show that the variational problem (91)–(93) is equivalent to the following integral system:

$$\begin{aligned} v_1(x, t) &= c_1^i(0, x, t) \left[v_2(0, \tau_1(0)) + [S_i u]_1(\tau_1(0)) \right] \\ &\quad - \int_0^x d_1^i(\xi, x, t) \left(b_{12} v_2 + [a_3 Jv] - [Q_i u]_1 \right) (\xi, \tau_1(\xi)) d\xi, \end{aligned} \quad (97)$$

$$\begin{aligned} v_2(x, t) &= c_2^i(1, x, t) \left[v_1(1, \tau_2(1)) + [S_i u]_2(\tau_2(1)) - 2r_1(\tau_2(1)) [a Jv](1, \tau_2(1)) \right] \\ &\quad - \int_1^x d_2^i(\xi, x, t) \left(b_{21} v_1 - [a_3 Jv] + [Q_i u]_2 \right) (\xi, \tau_2(\xi)) d\xi, \end{aligned} \quad (98)$$

where

$$d_j^i(\xi, x, t) = \frac{(-1)^j c_j^i(\xi, x, t)}{a(\xi, \tau_j(\xi, x, t))}.$$

In other words, any function $u \in \mathcal{C}^i$ satisfies (91)–(93) in a distributional sense if and only if u satisfies (97)–(98) pointwise.

To show the sufficiency, take an arbitrary sequence $u^l \in \mathcal{C}^{i+1}$ approaching u in \mathcal{C}^i and write $v^l = \partial_t^i u^l$. Then, taking into account (96), for any smooth function $\varphi : (0, 1) \times (0, T) \rightarrow \mathbb{R}$ with compact support we have

$$\begin{aligned} \langle (\partial_t - a\partial_x)v_1, \varphi \rangle &= \langle v_1, -\partial_t \varphi + \partial_x(a\varphi) \rangle = \lim_{l \rightarrow \infty} \langle v_1^l, -\partial_t \varphi + \partial_x(a\varphi) \rangle \\ &= \lim_{l \rightarrow \infty} \left\langle c_1^i(0, x, t) \left[v_2^l(0, \tau_1(0)) + [S_i u]_1(\tau_1(0)) \right] \right. \\ &\quad \left. - \int_0^x d_1^i(\xi, x, t) b_{12}(\xi, \tau_1(\xi)) v_2^l(\xi, \tau_1(\xi)) d\xi \right. \end{aligned}$$

$$\begin{aligned}
& - \int_0^x d_1^i(\xi, x, t) \left([a_3 J v^l] - [Q_i u]_1 \right) (\xi, \tau_1(\xi)) d\xi, -\partial_t \varphi + \partial_x(a\varphi) \Big\rangle \\
& = \lim_{l \rightarrow \infty} \left\langle - \left(b_{11} + i \frac{\partial_t a}{a} \right) v_1^l - b_{12} v_2^l - [a_3 J v^l] (x, t) + [Q_i u]_1 (x, t), \varphi \right\rangle \\
& = \left\langle - \left(b_{11} + i \frac{\partial_t a}{a} \right) v_1 - b_{12} v_2 - [a_3 J v] (x, t) + [Q_i u]_1 (x, t), \varphi \right\rangle.
\end{aligned}$$

Similarly we compute the generalized directional derivative $(\partial_t + a(x, t)\partial_x)u_2$. The sufficiency is thereby proved.

To show the necessity, assume that $u \in \mathcal{C}^i$ satisfies (91)–(93) in a distributional sense. Without destroying the equalities in \mathcal{D}' , we rewrite the system (91) in the form

$$\begin{aligned}
(\partial_t - a\partial_x) (c_1^i(1, x, t)v_1) &= c_1^i(1, x, t) \left(-b_{12}v_2 - [a_3 J v] (x, t) + [Q_i u]_1 (x, t) \right), \\
(\partial_t + a\partial_x) (c_2^i(0, x, t)v_2) &= c_2^i(0, x, t) \left(-b_{21}v_1 - [a_3 J v] (x, t) + [Q_i u]_2 (x, t) \right).
\end{aligned} \tag{99}$$

To prove that v satisfies (97)–(98) pointwise, we use the constancy theorem of distribution theory claiming that any distribution on an open set with zero generalized derivatives is a constant on any connected component of the set. Hence, the sums

$$v_1(x, t) + \int_0^x d_1^i(\xi, x, t) \left(b_{12} + [a_3 J v] - [Q_i u]_1 \right) (\xi, \tau_1(\xi)) d\xi$$

and

$$v_2(x, t) + \int_1^x d_2^i(\xi, x, t) \left(b_{21} + [a_3 J v] - [Q_i u]_2 \right) (\xi, \tau_2(\xi)) d\xi$$

are constants along the characteristics $\tau_1(\xi, x, t)$ and $\tau_2(\xi, x, t)$, respectively. Since they are continuous functions and the traces $v_1(0, t)$ and $v_2(1, t)$ are given by (93), it follows that v satisfies the system (97)–(98) as desired.

We are therefore reduced to prove that the function v satisfying the system (97)–(98) is continuously differentiable. To this end, for $i \geq 1$ we introduce linear bounded operators $B_i, A_i, D_i, R_i : \mathcal{C} \rightarrow \mathcal{C}$ by

$$\begin{aligned}
[B_i v](x, t) &= \left(c_1^i(0, x, t)v_2(0, \tau_1(0)), c_2^i(1, x, t)v_1(1, \tau_2(1)) \right) \\
[A_i v](x, t) &= \left(0, -2c_2^i(1, x, t)r_1(\tau_2(1))[a_3 J v](1, \tau_2(1)) \right),
\end{aligned} \tag{100}$$

$$\begin{aligned}
[D_i u](x, t) &= \left(- \int_0^x d_1^i(\xi, x, t) (b_{12}v_2 - [a_3 J v]) (\xi, \tau_1(\xi)) d\xi, \right. \\
&\quad \left. - \int_1^x d_2^i(\xi, x, t) (b_{21}v_1 - [a_3 J v]) (\xi, \tau_2(\xi)) d\xi \right), \\
[R_i u](x, t) &= \left(c_1^i(0, x, t)[S_i u]_1(\tau_1(0)) + \int_0^x d_1^i(\xi, x, t) [Q_i u]_1 (\xi, \tau_1(\xi)) d\xi, \right. \\
&\quad \left. c_2^i(1, x, t)[S_i u]_2(\tau_2(1)) - \int_1^x d_2^i(\xi, x, t) [Q_i u]_2 (\xi, \tau_2(\xi)) d\xi \right).
\end{aligned} \tag{101}$$

and rewrite (97)–(98) in the operator form

$$v = B_i v + A_i v + D_i v + R_i u. \quad (102)$$

Similarly to the base case of the induction, we will use the following equation for v (the analog of (79))

$$v = B_i v + (A_i^2 + A_i B_i + A_i D_i + D_i B_i + D_i A_i + D_i^2) v + (I + A_i + D_i) R_i u, \quad (103)$$

resulting from (102), and prove that it is uniquely solvable in $\tilde{\mathcal{C}}^1$. To this end, it is sufficient to show that

$$I - B_i \text{ is bijective from } \tilde{\mathcal{C}}^1 \text{ to } \tilde{\mathcal{C}}^1 \quad (104)$$

and that the operators

$$A_i^2, D_i^2, A_i D_i, D_i A_i, A_i B_i, \text{ and } D_i B_i \text{ map continuously } \mathcal{C} \text{ into } \tilde{\mathcal{C}}^1. \quad (105)$$

To prove (105), we follow a similar argument as in the proof of the corresponding property (68) in the base case of the induction. The only difference is that now in all the calculations involved we use c_j^i and d_j^i in place of c_j and d_j , respectively. In particular, to prove smoothing property (105) for A_i^2 , on the account of the definitions of A and A_i , we can follow the argument as in the proof of this property for A^2 and, hence, reduce the problem to the one for the operator JA_i where the operator A_i is defined by (100).

It remains to prove the bijectivity property (104). Again, following the same argument as in the base case, we actually have to show that the system

$$v(t) = [B'_i v](t) + g(t),$$

where the operator $B'_i \in \mathcal{L}(C_T(\mathbb{R}))$ is given by

$$[B'_i v](t) = c_1^i(0, 1, t) c_2^i(1, 0, \tau_1(0, 1, t)) v(\tau_2(1, 0, \tau_1(0, 1, t))), \quad (106)$$

is uniquely solvable in $C_T^1(\mathbb{R})$ for any $g \in C_T^1(\mathbb{R})$. The latter is true iff

$$I - B'_i \text{ is bijective from } C_T^1(\mathbb{R}) \text{ to } C_T^1(\mathbb{R}). \quad (107)$$

Obviously, (107) is true whenever

$$\|B'_i\|_{\mathcal{L}(C_T^1(\mathbb{R}))} < 1. \quad (108)$$

Now we show that (108) is a consequence of the contraction condition (30) with $l = i, i+1$. Similarly to the above we will again norm the space $C_T^1(\mathbb{R})$ with (82). The proof is completed by showing that there exist constants $\gamma_i < 1$ and $\beta_i < 1$ such that

$$\|B'_i v\|_\infty + \beta_i \left\| \frac{d}{dt} B'_i v \right\|_\infty \leq \gamma_i (\|v\|_\infty + \beta_i \|v'\|_\infty) \text{ for all } v \in C_T^1(\mathbb{R}). \quad (109)$$

By assumption (30), $\|B'_i\|_{\mathcal{L}(C_T(\mathbb{R}))} \leq q_i < 1$. Thus, the estimate (109) will be proved if we show that

$$\left\| \frac{d}{dt} B'_i v \right\|_{\infty} \leq \frac{\gamma_i - \tilde{q}_i}{\beta_i} \|v\|_{\infty} + \gamma_i i \|v'\|_{\infty} \quad \text{for all } v \in C_T^1(\mathbb{R}). \quad (110)$$

Since

$$\begin{aligned} \frac{d}{dt} [(B'_i v)(t)] &= \frac{d}{dt} \left[c_1^i(0, 1, t) c_2^i(1, 0, \tau_1(0, 1, t)) \right] v(\tau_2(1, 0, \tau_1(0, 1, t))) \\ &\quad + c_1^i(0, x, t) c_2^i(1, 0, \tau_1(0, 1, t)) \left[\partial_2 \tau_2(1, 0, \tau_1(0, 1, t)) \partial_t \tau_1(0, 1, t) \right]^i \\ &\quad \times v'(\tau_2(1, 0, \tau_1(0, 1, t))), \end{aligned}$$

we get

$$\left\| \frac{d}{dt} B'_i v \right\|_{\infty} \leq q'_i \|v\|_{\infty} + q_{i+1} \|v'\|_{\infty}, \quad (111)$$

where q'_i is given by (83). By assumption (30) we have $q_i < 1$ and $q_{i+1} < 1$. Fix γ_i such that $\max\{q_i, q_{i+1}\} < \gamma_i < 1$. Then choose β_i so small that

$$q'_i \leq \frac{\gamma_i - q_i}{\beta_i}.$$

Finally, (111) implies (110), what also finishes the proof of the bijectivity property of $I - B_i \in \mathcal{L}(\tilde{\mathcal{C}}^1)$.

5 Smooth dependence on the data: proof of Theorem 1.5

Here we establish smooth dependence of solutions to (1)–(3) on the coefficients of (1) and (3). With this aim in Section 1.1 we introduced a small parameter $\varepsilon \geq 0$ responsible for small perturbations of the coefficients. We therefore consider the perturbed problems (37)–(39) and (40)–(42).

In what follows we will use the following notation:

$$[B(\varepsilon)u](x, t) = \left(c_1^\varepsilon(0, x, t) u_2(0, \tau_1^\varepsilon(0)), c_2^\varepsilon(1, x, t) u_1(1, \tau_2^\varepsilon(1)) \right), \quad (112)$$

$$\begin{aligned} [A(\varepsilon)u](x, t) &= \left(2c_1^\varepsilon(0, x, t) a^\varepsilon(0, \tau_1^\varepsilon(0)) r_0^\varepsilon(\tau_1^\varepsilon(0)) [G(\varepsilon)u](\tau_1^\varepsilon(0)), \right. \\ &\quad \left. - 2c_2^\varepsilon(1, x, t) a^\varepsilon(1, \tau_2^\varepsilon(1)) r_1(\tau_2^\varepsilon(1)) [F(\varepsilon)u](1, \tau_2^\varepsilon(1)) \right), \end{aligned} \quad (113)$$

$$\begin{aligned} [D(\varepsilon)u](x, t) &= \left(- \int_0^x d_1^\varepsilon(\xi, x, t) (b_{12}^\varepsilon u_2 - [a_3^\varepsilon F(\varepsilon)u](\xi, \tau_1^\varepsilon(\xi))) d\xi, \right. \\ &\quad \left. - \int_1^x d_2^\varepsilon(\xi, x, t) (b_{21}^\varepsilon u_1 - [a_3^\varepsilon F(\varepsilon)u](\xi, \tau_2^\varepsilon(\xi))) d\xi \right), \end{aligned} \quad (114)$$

$$[G(\varepsilon)u](t) = [Iu](t) + N(\varepsilon)u, \quad (115)$$

$$[F(\varepsilon)u](x, t) = [G(\varepsilon)u](t) + [J(\varepsilon)u](x, t), \quad (116)$$

$$[J(\varepsilon)u](x, t) = \int_0^x \frac{u_1(\xi, t) - u_2(\xi, t)}{2a^\varepsilon(\xi, t)} d\xi, \quad (117)$$

$$N(\varepsilon)u = \frac{1}{C} \int_0^T \left(\frac{u_1(0, t) - u_2(0, t)}{2} - a^\varepsilon(0, t)r_0^\varepsilon(t)[Iu](t) \right) dt, \quad (118)$$

$$c_j^{i\varepsilon}(\xi, x, t) = \exp \int_x^\xi (-1)^j \left(\frac{b_{jj}^\varepsilon}{a^\varepsilon} - i \frac{\partial_t a^\varepsilon}{a^{\varepsilon 2}} \right) (\eta, \tau_j^\varepsilon(\eta, x, t)) d\eta, \quad (119)$$

$$d_j^{i\varepsilon}(\xi, x, t) = \frac{(-1)^j c_j^{i\varepsilon}(\xi, x, t)}{a^\varepsilon(\xi, \tau_j^\varepsilon(\xi, x, t))}, \quad (120)$$

$$[B_i(\varepsilon)u](x, t) = \left(c_1^{i\varepsilon}(0, x, t)u_2(0, \tau_1^\varepsilon(0)), c_2^{i\varepsilon}(1, x, t)u_1(1, \tau_2^\varepsilon(1)) \right), \quad (121)$$

$$[A_i(\varepsilon)v](x, t) = \left(0, -2c_2^{i\varepsilon}(1, x, t)r_1^\varepsilon(\tau_2(1))[a^\varepsilon J(\varepsilon)v](1, \tau_2^\varepsilon(1)) \right), \quad (122)$$

$$\begin{aligned} [D_i(\varepsilon)u](x, t) = & \left(- \int_0^x d_1^{i\varepsilon}(\xi, x, t) (b_{12}^\varepsilon v_2 - [a_3^\varepsilon J(\varepsilon)v](\xi, \tau_1^\varepsilon(\xi))) d\xi, \right. \\ & \left. - \int_1^x d_2^{i\varepsilon}(\xi, x, t) (b_{21}^\varepsilon v_1 - [a_3^\varepsilon J(\varepsilon)v](\xi, \tau_2^\varepsilon(\xi))) d\xi \right), \end{aligned} \quad (123)$$

where $i = 1, 2, \dots$ and $\tau_j^\varepsilon(\xi, x, t)$ is the solution to the initial value problem (20) with a^ε in place of a .

From the definitions of c_j^ε and τ_j^ε and the regularity assumption (43) one can easily derive the bounds (needed to prove Lemma 5.1 below)

$$\begin{aligned} \|c_1^{i\varepsilon'}(0, x, t) - c_1^{i\varepsilon''}(0, x, t)\|_{\mathcal{C}} &= O(|\varepsilon' - \varepsilon''|), \\ \|c_2^{i\varepsilon'}(1, x, t) - c_2^{i\varepsilon''}(1, x, t)\|_{\mathcal{C}} &= O(|\varepsilon' - \varepsilon''|), \\ \|\tau_1^{\varepsilon'}(0) - \tau_1^{\varepsilon''}(0)\|_{\mathcal{C}} &= O(|\varepsilon' - \varepsilon''|), \quad \|\tau_2^{\varepsilon'}(1) - \tau_2^{\varepsilon''}(1)\|_{\mathcal{C}} = O(|\varepsilon' - \varepsilon''|), \end{aligned} \quad (124)$$

being true for $j = 1, 2$, all $\varepsilon', \varepsilon'' < 1$ and all nonnegative integers $i \leq k$.

In this section without restriction of generality we will work under the assumptions (4), (5), (30), and (43). A similar argument works if we replace (30) by one of the conditions (31), (32), and (33). One can easily check that in the case of t -independent a the following is true: if one of the conditions (26) and (27) is fulfilled, then one of the conditions (30), (31), (32), and (33) is fulfilled as well. This fact together with Theorem 1.2, Theorem 1.4 (ii), and Theorem 1.5 (i) entail Theorem 1.5 (ii).

To state (i), it suffices to prove the smooth dependence result for u^ε on ε : the value of ε_0 can be chosen so small that for all $\varepsilon \leq \varepsilon_0$ there exists a unique weak solution to (40)–(42) which belongs to C_T^k , and the map $\varepsilon \in [0, \varepsilon_0] \mapsto u^\varepsilon \in \mathcal{C}^{k-\gamma-1}$ is C^γ -smooth for all non-negative integers $\gamma \leq k - 1$.

Note that conditions (4), (5), and (30) are stable with respect to small perturbations of all functions contributing into them. Fix ε_0 so small that those conditions are fulfilled for all $\varepsilon \leq \varepsilon_0$ with $a(x, t)$, $a_1(x, t)$, $a_2(x, t)$, and $r_0(t)$ replaced by $a(x, t, \varepsilon)$, $a_1(x, t, \varepsilon)$, $a_2(x, t, \varepsilon)$, and $r_0(t, \varepsilon)$, respectively. Then for all $\varepsilon \leq \varepsilon_0$ all conditions of Theorems 1.2

and 1.4 are fulfilled. It follows that, given $\varepsilon \leq \varepsilon_0$, there exists a weak solution to (40)–(42) which belongs to C_T^k . The uniqueness of the weak solution will follow from the bijectivity property of the operator $I - C(\varepsilon) - A(\varepsilon) - D(\varepsilon)$.

Lemma 5.1 (i) *There is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the operator $I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)$ is bijective from \mathcal{C}^1 to \mathcal{C}^1 and satisfies the estimate*

$$\|(I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{C}^1)} = O(1) \quad (125)$$

uniformly in $\varepsilon \leq \varepsilon_0$.

(ii) *Given $i \leq k - 1$, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the operator $I - B_i(\varepsilon) - A_i(\varepsilon) - D_i(\varepsilon)$ is bijective from \mathcal{C}^1 to \mathcal{C}^1 and satisfies the estimate*

$$\|(I - B_i(\varepsilon) - A_i(\varepsilon) - D_i(\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{C}^1)} = O(1) \quad (126)$$

uniformly in $\varepsilon \leq \varepsilon_0$.

(iii) *Given $i \leq k$, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the operator $I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)$ is bijective from \mathcal{C}^i to \mathcal{C}^i and satisfies the estimate*

$$\|(I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{C}^i)} = O(1) \quad (127)$$

uniformly in $\varepsilon \leq \varepsilon_0$.

Proof. (i) Recall that within the assumptions of Theorem 1.5 we meet all conditions of Theorem 1.2 whenever $\varepsilon \leq \varepsilon_0$. Hence, given $\varepsilon \leq \varepsilon_0$, the operator of the problem (40)–(42) or, the same, the operator $I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)$ is Fredholm from \mathcal{C} to \mathcal{C} . If $\dim \ker(I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)) = 0$, then it is bijective from \mathcal{C} to \mathcal{C} . From Theorem 1.4 it follows that $I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)$ is surjective from \mathcal{C}^1 onto \mathcal{C}^1 . Hence, it remains to show the injectivity of $I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)$ from \mathcal{C} to \mathcal{C} , what is the same, from \mathcal{C}^1 onto \mathcal{C}^1 .

Suppose, on the contrary, that there exist sequences $\varepsilon_n \rightarrow_{n \rightarrow \infty} 0$ and $u^n \in \mathcal{C}^1$ such that

$$\|u^n\|_{\mathcal{C}^1} = 1 \quad (128)$$

and

$$u^n = B(\varepsilon_n)u^n + A(\varepsilon_n)u^n + D(\varepsilon_n)u^n. \quad (129)$$

Hence,

$$u^n = B(\varepsilon_n)u^n + (A(\varepsilon_n) + D(\varepsilon_n))(B(\varepsilon_n) + A(\varepsilon_n) + D(\varepsilon_n))u^n. \quad (130)$$

Due to the choice of ε_0 , the operators $I - B(\varepsilon) \in \mathcal{L}(\mathcal{C})$ and $I - B(\varepsilon) \in \mathcal{L}(\mathcal{C}^1)$ are invertible and satisfy the estimates

$$\|(I - B(\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{C})} = O(1), \quad \|(I - B(\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{C}^1)} = O(1), \quad (131)$$

being uniform in $\varepsilon \leq \varepsilon_0$. Thus, we are able to rewrite (130) as follows:

$$\begin{aligned}
u^n &= (I - B(\varepsilon_n))^{-1} \left(A(\varepsilon_n) + D(\varepsilon_n) \right) \left(B(\varepsilon_n) + A(\varepsilon_n) + D(\varepsilon_n) \right) u^n \\
&= (I - B)^{-1} (A + D) (B + A + D) u^n \\
&+ \left[(I - B(\varepsilon_n))^{-1} \left(A(\varepsilon_n) + D(\varepsilon_n) \right) \left(B(\varepsilon_n) + A(\varepsilon_n) + D(\varepsilon_n) \right) \right. \\
&- \left. (I - B)^{-1} (A + D) (B + A + D) \right] u^n.
\end{aligned} \tag{132}$$

Let us show that the map

$$\begin{aligned}
\varepsilon \in [0, \varepsilon_0) &\mapsto (I - B(\varepsilon))^{-1} (A(\varepsilon) + D(\varepsilon)) (B(\varepsilon) + A(\varepsilon) + D(\varepsilon)) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C}) \\
&\text{is locally Lipschitz continuous.}
\end{aligned} \tag{133}$$

First we show that the map

$$\varepsilon \in [0, \varepsilon_0) \mapsto (I - B(\varepsilon))^{-1} \in \mathcal{L}(\mathcal{C}^1, \mathcal{C}) \text{ is locally Lipschitz continuous.} \tag{134}$$

Indeed, take $\varepsilon', \varepsilon'' \leq \varepsilon_0$, $f \in \mathcal{C}^1$, and $u', u'' \in \mathcal{C}^1$ such that

$$u' = B(\varepsilon')u' + f, \quad u'' = B(\varepsilon'')u'' + f.$$

Hence,

$$u' - u'' = (B(\varepsilon') - B(\varepsilon''))u' + B(\varepsilon'')(u' - u''),$$

or, the same,

$$u' - u'' = (I - B(\varepsilon''))^{-1} (B(\varepsilon') - B(\varepsilon''))u'. \tag{135}$$

Since $u', u'' \in \mathcal{C}^1$, then on the account of (121), we have

$$\begin{aligned}
(B(\varepsilon') - B(\varepsilon''))u' &= \left(\left(c_1^{\varepsilon'}(0, x, t) - c_1^{\varepsilon''}(0, x, t) \right) u_2'(0, \tau_1^{\varepsilon'}(0)) \right. \\
&+ c_1^{\varepsilon''}(0, x, t) \int_0^1 \partial_2 u_2' \left(0, \alpha \tau_1^{\varepsilon'}(0) + (1 - \alpha) \tau_1^{\varepsilon''}(0) \right) d\alpha \left(\tau_1^{\varepsilon'}(0) - \tau_1^{\varepsilon''}(0) \right), \\
&\left(c_2^{\varepsilon'}(1, x, t) - c_2^{\varepsilon''}(1, x, t) \right) u_1'(1, \tau_1^{\varepsilon'}(1)) \\
&+ c_2^{\varepsilon''}(1, x, t) \int_0^1 \partial_2 u_1' \left(0, \alpha \tau_2^{\varepsilon'}(1) + (1 - \alpha) \tau_2^{\varepsilon''}(1) \right) d\alpha \left(\tau_2^{\varepsilon'}(1) - \tau_2^{\varepsilon''}(1) \right) \Big).
\end{aligned} \tag{136}$$

Thus, the equation (135) is well defined in \mathcal{C} . Note that, by (131), there is a constant $c > 0$ not depending on ε' and f such that

$$\|u'\|_{\mathcal{C}^1} \leq c \|f\|_{\mathcal{C}^1}. \tag{137}$$

Now, taking into account the definition (112) and the bounds (124), (131), and (137), from (136) we derive the estimate

$$\|u' - u''\|_{\mathcal{C}} \leq c |\varepsilon' - \varepsilon''| \|f\|_{\mathcal{C}^1}$$

with a new constant c independent of ε' , ε'' , and f . Therewith (134) is proved.

To finish with (133), we take into account the definitions (112)–(114) of the operators $B(\varepsilon)$, $A(\varepsilon)$, $D(\varepsilon)$ and get $B(\varepsilon)$, $A(\varepsilon)$, $D(\varepsilon) \in \mathcal{L}(\mathcal{C}^1)$ as well as $B(\varepsilon)$, $A(\varepsilon)$, $D(\varepsilon) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$ are locally Lipschitz continuous in ε . Hence, we have $(A(\varepsilon) + D(\varepsilon))(B(\varepsilon) + A(\varepsilon) + D(\varepsilon)) \in \mathcal{L}(\mathcal{C}^1)$ and also the map $\varepsilon \in [0, \varepsilon_0) \mapsto (A(\varepsilon) + D(\varepsilon))(B(\varepsilon) + A(\varepsilon) + D(\varepsilon)) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$ is locally Lipschitz continuous. This finishes the proof of (133).

Now, returning to (132), we conclude that the second summand in the right-hand side tends to zero in \mathcal{C} as $n \rightarrow \infty$, while a subsequence of $(I - B)^{-1}(A + D)(B + A + D)u^n$ (the first summand) converges in \mathcal{C} . Therefore, a subsequence of u^n (further denoted by u^n again) converges to a function $u \in \mathcal{C}$. Our aim now is to show that passing to the limit in (129) gives

$$u = (B + A + D)u, \quad (138)$$

the equality being true in \mathcal{C} . This means that $u \in \mathcal{K}_u$, where \mathcal{K}_u the vector space of all weak solutions to (16)–(18) with $f = 0$. Hence, by Theorem 1.4, the function u has \mathcal{C}^1 regularity. On the other hand, due to (128), $\|u\|_{\mathcal{C}^1} = 1$, a contradiction with $u \in \mathcal{K}_u$ $u \in \mathcal{C}^1$, and $\dim \mathcal{K}_u = 0$.

We are left with proving (138). Above we showed that u^n and, hence, the right-hand side of (129) converges in \mathcal{C} . Thus, we are done if we prove that

$$B(\varepsilon_n)u^n \rightarrow Bu, \quad A(\varepsilon_n)u^n \rightarrow Au, \quad D(\varepsilon_n)u^n \rightarrow Du \quad \text{in } \mathcal{C} \text{ as } n \rightarrow \infty.$$

Let us prove the first convergence (similar proof is true for the other two). We have

$$B(\varepsilon_n)u^n - Bu = (B(\varepsilon_n) - B)u^n + B(u^n - u).$$

The first summand in the right-hand side tends to zero in \mathcal{C} thanks to (128) and the locally Lipschitz continuity of the map $\varepsilon \in [0, \varepsilon_0) \mapsto B(\varepsilon) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$ and the second one – due to the convergency of u^n in \mathcal{C} . In this way we reach the desired convergence. Finally, passing to the limit as $n \rightarrow \infty$ in (129) gives (138).

We therefore proved that the problem (40)–(42) is uniquely solvable in \mathcal{C}^1 for each $\varepsilon \leq \varepsilon_0$. To finish the proof of the claim (i), it remains to prove the estimate (125). If this is not the case, then there exist sequences $\varepsilon_n \rightarrow_{n \rightarrow \infty} 0$ and $u^n \in \mathcal{C}^1$ satisfying (128) and

$$u^n - B(\varepsilon_n)u^n - A(\varepsilon_n)u^n - D(\varepsilon_n)u^n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ in } \mathcal{C}^1. \quad (139)$$

We proceed similarly to the above with (129) in place of (80), up to getting a contradiction.

(ii) Fix an arbitrary $i \leq k - 1$. Theorem 1.2 states the Fredholmness of $I - B_i(\varepsilon) - A_i(\varepsilon) - D_i(\varepsilon)$ from \mathcal{C}^1 to \mathcal{C} for all sufficiently small ε , while Theorem 1.4 strengthens this result to the Fredholmness of $I - B_i(\varepsilon) - A_i(\varepsilon) - D_i(\varepsilon)$ from \mathcal{C}^1 to \mathcal{C}^1 . This means that the desired statement will be proved whenever we show the injectivity of $I - B_i(\varepsilon) - A_i(\varepsilon) - D_i(\varepsilon)$ from \mathcal{C}^1 to \mathcal{C}^1 .

Assume, conversely, that there exist sequences $\varepsilon_n \rightarrow 0$ and $u^n \in \mathcal{C}^1$ fulfilling (128) and

$$u^n = B_i(\varepsilon_n)u^n + A_i(\varepsilon_n)u^n + D_i(\varepsilon_n)u^n. \quad (140)$$

Due to the choice of ε_0 , the operators $I - B_i(\varepsilon) \in \mathcal{L}(\mathcal{C})$ and $I - B_i(\varepsilon) \in \mathcal{L}(\mathcal{C}^1)$ are invertible and satisfy the estimates

$$\|(I - B_i(\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{C})} = O(1), \quad \|(I - B_i(\varepsilon))^{-1}\|_{\mathcal{L}(\mathcal{C}^1)} = O(1) \quad (141)$$

uniformly in $\varepsilon \leq \varepsilon_0$. This entails, in particular, that there are constants $c > 0$ and ε_0 such that for all $\varepsilon \leq \varepsilon_0$ and $f \in \mathcal{C}^1$ the continuously differentiable solution to the equation $u = B_i(\varepsilon)u + f$ satisfies the apriory estimate

$$\|u\|_{\mathcal{C}^1} \leq c\|f\|_{\mathcal{C}^1}. \quad (142)$$

Moreover, from (140) we get

$$\begin{aligned} u^n &= (I - B_i)^{-1} (A_i + D_i) (B_i + A_i + D_i) u^n \\ &+ \left[(I - B_i(\varepsilon_n))^{-1} (A_i(\varepsilon_n) + D_i(\varepsilon_n)) (B_i(\varepsilon_n) + A_i(\varepsilon_n) + D_i(\varepsilon_n)) \right. \\ &\quad \left. - (I - B_i)^{-1} (A_i + D_i) (B_i + A_i + D_i) \right] u^n, \end{aligned} \quad (143)$$

where $B_i = B_i(0)$, $A_i = A_i(0)$, $D_i = D_i(0)$.

Next, we need that the maps

$$\varepsilon \in [0, \varepsilon_0) \mapsto (I - B_i(\varepsilon_n))^{-1} \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$$

and

$$\varepsilon \in [0, \varepsilon_0) \mapsto (A_i(\varepsilon) + D_i(\varepsilon)) (B_i(\varepsilon) + A_i(\varepsilon) + D_i(\varepsilon)) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$$

are locally Lipschitz continuous. Proceeding analogously to the proof of (134), we state that the former follows from the smoothness assumptions on the data as well as from the estimates (141), (142), and (124), while the latter is a consequence of the facts that $B_i(\varepsilon), A_i(\varepsilon), D_i(\varepsilon) \in \mathcal{L}(\mathcal{C}^1)$ and that $B_i(\varepsilon), A_i(\varepsilon), D_i(\varepsilon) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$ are locally Lipschitz continuous in ε . This entails also the desired property $(A_i(\varepsilon) + D_i(\varepsilon)) (B_i(\varepsilon) + A_i(\varepsilon) + D_i(\varepsilon)) \in \mathcal{L}(\mathcal{C}^1)$.

Now, accordingly to (143), a subsequence of u^n (below denoted by u^n again) converges in \mathcal{C} to a function $u \in \mathcal{C}$. To get a contradiction, similarly to the above, it remains to show that u satisfies the equation

$$u = (B_i + A_i + D_i) u \quad (144)$$

in \mathcal{C} . We derive the latter from (140) applying the convergency of u^n to u in \mathcal{C} as well as the locally Lipschitz continuity of the maps $\varepsilon \in [0, \varepsilon_0) \mapsto B_i(\varepsilon) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$, $\varepsilon \in [0, \varepsilon_0) \mapsto A_i(\varepsilon) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$, and $\varepsilon \in [0, \varepsilon_0) \mapsto D_i(\varepsilon) \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$. Equality (144) means that $u \in \ker(I - B_i - A_i - D_i)$ and, hence, by Theorem 1.4, the function u has \mathcal{C}^1 regularity. On the other hand, due to (128), $\|u\|_{\mathcal{C}^1} = 1$, a contradiction with $u \in \ker(I - B_i - A_i - D_i)$, $u \in \mathcal{C}^1$, and

$$\dim \ker(I - B_i - A_i - D_i) = 0. \quad (145)$$

Briefly speaking, the latter follows from the facts that $\mathcal{K}_u = 0$ and that any solution to (138) has \mathcal{C}^k -regularity, what, on the account of the proof of Theorem 1.4, necessarily leads to the unique solvability of (144) in \mathcal{C} for every $i \leq k$.

To prove (145) in details, we will use the induction on i . To prove the *base case* $i = 1$, given $f \in \mathcal{C}^k$, let u be the unique solution to the equation

$$u = (B + A + D)u + Rf. \quad (146)$$

By Theorem 1.4, $u \in \mathcal{C}^k$. Taking into account the proof of Theorem 1.4, the unique solvability of (146) in \mathcal{C}^1 is equivalent to the unique solvability in \mathcal{C} of the system

$$\begin{aligned} u &= (B + A + D)u + Rf, \\ v &= (B_1 + A_1 + D_1)v + R_1u \end{aligned} \quad (147)$$

with respect to (u, v) . Here $v = \partial_t u$. As $\mathbb{K}_u = 0$, the first equation in (147) is uniquely solvable in \mathcal{C} , and we have

$$u = [I - B - A - D]^{-1} Rf.$$

Now it remains to note that the system (147) is uniquely solvable in \mathcal{C} iff the operator $I - B_1 - A_1 - D_1 \in \mathcal{L}(\mathcal{C})$ is bijective from \mathcal{C} to \mathcal{C} . The base case is therewith proved. Given $2 \leq i \leq k - 1$, assume that $\dim \ker (I - B_j - A_j - D_j) = 0$ in \mathcal{C} for all $2 \leq j \leq i - 1$ (*induction assumption*) and prove that $\dim \ker (I - B_i - A_i - D_i) = 0$ in \mathcal{C} (*induction step*). Since the solution u to (146) belongs to \mathcal{C}^k , then it satisfies the equation

$$w = (B_i + A_i + D_i)w + R_i u, \quad (148)$$

where $w = \partial_t^i u$ and the operator R_i given by (101), (94), and (95) is linear operator of $u, \partial_t u, \dots, \partial_t^{i-1} u$. Note that $u, \partial_t u, \dots, \partial_t^{i-1} u$ are the continuous functions uniquely determined due to the induction assumption. Again, because of (146) is uniquely solvable in \mathcal{C}^k and $[R_i u](x, t)$ is a known continuous function, the equation (146) is uniquely solvable with respect to w in \mathcal{C} iff $I - B_i - A_i - D_i \in \mathcal{L}(\mathcal{C})$ is bijective from \mathcal{C} to \mathcal{C} . The induction step is proved.

The proof of the estimate (126) follows the same line as the proof of (125), what finishes the proof of this claim.

Claim (iii) easily follows from Claims (i) and (ii) and the proof of Theorem 1.4. \square

Write $B_0(\varepsilon) = B(\varepsilon)$, $A_0(\varepsilon) = A(\varepsilon)$, and $D_0(\varepsilon) = D(\varepsilon)$.

Lemma 5.2 (i) *The map*

$$\varepsilon \in [0, \varepsilon_0) \mapsto [I - B_i(\varepsilon) - A_i(\varepsilon) - D_i(\varepsilon)]^{-1} \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$$

is locally Lipschitz continuous for all non-negative integers $i \leq k - 1$.

(ii) *The map*

$$\varepsilon \in [0, \varepsilon_0) \mapsto [I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)]^{-1} \in \mathcal{L}(\mathcal{C}^{i+1}, \mathcal{C}^i)$$

is locally Lipschitz continuous for all non-negative integers $i \leq k - 1$.

Claim (i) may be proved in much the same way as the proof of (134) and uses now Lemma 5.1 (i)–(ii). Claim (ii) may be proved by induction on i using Lemma 5.1 (iii) and Lemma 5.2 (i).

To finish the proof of Theorem 1.5, what is left is to show that the map $\varepsilon \in [0, \varepsilon_0) \mapsto u^\varepsilon \in \mathcal{C}^{k-\gamma-1}$ is C^γ -smooth for all non-negative integers $\gamma \leq k-1$. The proof of this statement will be by induction on k .

Base case: The map $\varepsilon \in [0, \varepsilon_0) \mapsto u^\varepsilon \in \mathcal{C}$ is continuous. The claim follows from Lemma 5.2 (i) with $i = 0$.

Let $k \geq 2$.

Induction assumption: The map $\varepsilon \in [0, \varepsilon_0) \mapsto u^\varepsilon \in \mathcal{C}^{k-\gamma-2}$ is C^γ -smooth for all non-negative integers $\gamma \leq k-2$.

Induction step: The map $\varepsilon \in [0, \varepsilon_0) \mapsto u^\varepsilon \in \mathcal{C}^{k-\gamma-1}$ is C^γ -smooth for all non-negative integers $\gamma \leq k-1$. To prove the induction step, again we use induction but this time on γ .

Checking the *base case* $\gamma = 0$ we have to show that $v^\varepsilon = \partial_t^{k-1} u^\varepsilon$ depends continuously on ε . Since $u^\varepsilon \in \mathcal{C}^k$, we have $v^\varepsilon \in \mathcal{C}^1$. Hence v^ε fulfills the system (88) with $i = k-1$ pointwise where all the data involved are not fixed at $\varepsilon = 0$ now, but are ε -dependent. The latter is equivalent to the integral system (97)–(98) with $i = k-1$ and with ε -dependent coefficients involved or, the same, to the operator equation

$$v^\varepsilon = B_{k-1}(\varepsilon)v^\varepsilon + A_{k-1}(\varepsilon)v^\varepsilon + D_{k-1}(\varepsilon)v^\varepsilon + R_{k-1}(\varepsilon)u^\varepsilon, \quad (149)$$

where

$$\begin{aligned} [R_i(\varepsilon)u](x, t) &= \left(c_1^{i\varepsilon}(0, x, t)[S_i(\varepsilon)u^\varepsilon]_1(\tau_1^\varepsilon(0)) + \int_0^x d_1^{i\varepsilon}(\xi, x, t)[Q_i(\varepsilon)u^\varepsilon]_1(\xi, \tau_1^\varepsilon(\xi)) d\xi, \right. \\ &\quad \left. c_2^{i\varepsilon}(1, x, t)[S_i(\varepsilon)u^\varepsilon]_2(\tau_2^\varepsilon(1)) - \int_1^x d_2^{i\varepsilon}(\xi, x, t)[Q_i(\varepsilon)u^\varepsilon]_2(\xi, \tau_2^\varepsilon(\xi)) d\xi \right), \\ [Q_i(\varepsilon)u^\varepsilon](x, t) &= \left(f_{1i}^\varepsilon(x, t, u^\varepsilon, \partial_t u^\varepsilon, \dots, \partial_t^{i-1} u^\varepsilon) - [P_i(\varepsilon)u^\varepsilon](x, t), \right. \\ &\quad \left. f_{2i}^\varepsilon(x, t, u^\varepsilon, \partial_t u^\varepsilon, \dots, \partial_t^{i-1} u^\varepsilon) - [P_i(\varepsilon)u^\varepsilon](x, t) \right), \\ [S_i(\varepsilon)u^\varepsilon](t) &= \left(2\partial_t^i(a^\varepsilon(0, t)r_0(t))[G(\varepsilon)u^\varepsilon](t) + \partial_t^{i-1}(a^\varepsilon(0, t)r_0^\varepsilon(t)(u_1^\varepsilon(0, t) + u_2^\varepsilon(0, t))), \right. \\ &\quad - 2\partial_t^i(a^\varepsilon(1, t)r_1^\varepsilon(t))[G(\varepsilon)u^\varepsilon](t) - \partial_t^{i-1}[a^\varepsilon(1, t)r_1^\varepsilon(t)(u_1^\varepsilon(0, t) + u_2^\varepsilon(0, t))] \\ &\quad \left. - \partial_t^{i-1} \int_0^x \partial_t \left(\frac{a^\varepsilon(1, t)r_1^\varepsilon(t)}{a^\varepsilon(\xi, t)} \right) (u_1^\varepsilon(\xi, t) - u_2^\varepsilon(\xi, t)) d\xi \right), \\ [P_i(\varepsilon)u^\varepsilon](x, t) &= \partial_t^i a_3^\varepsilon[Iu^\varepsilon](t) + \frac{1}{2}\partial_t^{i-1}[a_3^\varepsilon(u_1^\varepsilon(0, t) + u_2^\varepsilon(0, t))] \\ &\quad + \partial_t^{i-1} \int_0^x \partial_t \left(\frac{a_3^\varepsilon(x, t)}{a^\varepsilon(\xi, t)} \right) \frac{u_1^\varepsilon(\xi, t) - u_2^\varepsilon(\xi, t)}{2} d\xi \end{aligned} \quad (150)$$

and the operators $B_i(\varepsilon)$, $A_i(\varepsilon)$, $D_i(\varepsilon)$, $G(\varepsilon) \in \mathcal{C} \mapsto \mathcal{C}$ are defined by (121), (122), (123), (115), and (116). The continuously differentiable functions f_{ji}^ε are defined by the same rules as f_{ji} but only with ε -perturbed coefficients involved. On the account of Lemma 5.1 (ii) and the fact that $[R_{k-1}(\varepsilon)u^\varepsilon](x, t) \in \mathcal{C}^1$ for each $\varepsilon \leq \varepsilon_0$, we are able to rewrite (149)

in \mathcal{C}^1 as

$$v^\varepsilon = [I - B_{k-1}(\varepsilon) - A_{k-1}(\varepsilon)v^\varepsilon - D_{k-1}(\varepsilon)]^{-1} R_{k-1}(\varepsilon)u^\varepsilon.$$

Recall that R_{k-1} is a certain linear operator of $u^\varepsilon, \partial_t u^\varepsilon, \partial_t^{k-2} u^\varepsilon$. First we state that the map $\varepsilon \in [0, \varepsilon_0) \mapsto R_{k-1}(\varepsilon) \in \mathcal{L}(\mathcal{C}^1)$ is locally Lipschitz continuous, what follows from the definition of $R_{k-1}(\varepsilon)$ and the regularity assumptions on the initial data. Further we use the induction assumption on k allowing to conclude that $u^\varepsilon, \partial_t u^\varepsilon, \partial_t^{k-2} u^\varepsilon$ are locally Lipschitz continuous in ε . Finally, applying Lemma 5.2 (ii) with $i = k-1$ entails the locally Lipschitz continuity of the map $\varepsilon \in [0, \varepsilon_0) \mapsto [I - B_{k-1}(\varepsilon) - A_{k-1}(\varepsilon) - D_{k-1}(\varepsilon)]^{-1} \in \mathcal{L}(\mathcal{C}^1, \mathcal{C})$, what finishes the proof of the base case $\gamma = 0$.

Now for an arbitrary fixed $1 \leq \gamma \leq k-1$ assume that the map $\varepsilon \in [0, \varepsilon_0) \mapsto u^\varepsilon \in C_T^{k-\gamma}$ is $\mathcal{C}^{\gamma-1}$ -smooth in ε (*induction assumption*) and prove that the map $\varepsilon \in [0, \varepsilon_0) \mapsto u^\varepsilon \in C_T^{k-\gamma-1}$ is C^γ -smooth in ε (*induction step*). Let w^ε be a classical solution to the problem

$$\begin{aligned} (\partial_t - a^\varepsilon \partial_x) w_1^\varepsilon &= -b_{11}^\varepsilon w_1^\varepsilon - b_{12}^\varepsilon w_2^\varepsilon - a_3 F(\varepsilon) u^\varepsilon + \tilde{F}(\varepsilon) u^\varepsilon + \partial_\varepsilon^{\gamma-1} (\partial_\varepsilon a \partial_x v_1^\varepsilon) \\ (\partial_t + a^\varepsilon \partial_x) w_2^\varepsilon &= -b_{21}^\varepsilon w_1^\varepsilon - b_{22}^\varepsilon w_2^\varepsilon - a_3 F(\varepsilon) u^\varepsilon + \tilde{F}(\varepsilon) u^\varepsilon - \partial_\varepsilon^{\gamma-1} (\partial_\varepsilon a \partial_x v_2^\varepsilon), \end{aligned} \quad (151)$$

$$w_j^\varepsilon(x, t) = w_j^\varepsilon(x, t+T), \quad x \in [0, 1], \quad j = 1, 2, \quad (152)$$

$$\begin{aligned} w_1^\varepsilon(0, t) &= w_2^\varepsilon(0, t) + 2a^\varepsilon(0, t)r_0^\varepsilon(t) [Gw^\varepsilon](t) + 2\partial_\varepsilon^{\gamma-1} [\partial_\varepsilon (a^\varepsilon(0, t)r_0^\varepsilon(t)) [Gw^\varepsilon](t)] \\ w_2^\varepsilon(1, t) &= w_1^\varepsilon(1, t) - 2a^\varepsilon(1, t)r_1^\varepsilon(t) [Fw^\varepsilon](1, t) - 2\partial_\varepsilon^{\gamma-1} [\partial_\varepsilon (a^\varepsilon(1, t)r_1^\varepsilon(t)) [Fw^\varepsilon](1, t)] \end{aligned} \quad (153)$$

or, the same, the problem

$$w^\varepsilon = B(\varepsilon)w^\varepsilon + A(\varepsilon)w^\varepsilon + D(\varepsilon)w^\varepsilon + Q_{\gamma-1}(\varepsilon)u^\varepsilon + \tilde{R}_{\gamma-1}(\varepsilon)u^\varepsilon, \quad (154)$$

where

$$\begin{aligned} [Q_{\gamma-1}(\varepsilon)u^\varepsilon](x, t) &= \left(\int_0^x d_1^\varepsilon(\xi, x, t) [\partial_\varepsilon^{\gamma-1} (\partial_\varepsilon a^\varepsilon \partial_x u_1^\varepsilon)](\xi, \tau_1^\varepsilon(\xi)) d\xi, \right. \\ &\quad \left. \int_1^x d_2^\varepsilon(\xi, x, t) [\partial_\varepsilon^{\gamma-1} (\partial_\varepsilon a^\varepsilon \partial_x u_2^\varepsilon)](\xi, \tau_2^\varepsilon(\xi)) d\xi \right), \\ [\tilde{R}_{\gamma-1}(\varepsilon)u^\varepsilon]_1(x, t) &= 2c_1^\varepsilon(0, x, t) \partial_\varepsilon^{\gamma-1} [\partial_\varepsilon (a^\varepsilon(0, t)r_0^\varepsilon(t)) [Gu^\varepsilon](t)] \\ &\quad + \int_0^x d_1^\varepsilon(\xi, x, t) [\tilde{F}_{\gamma-1}(\varepsilon)u^\varepsilon](\xi, \tau_1(\xi)) d\xi, \\ [\tilde{R}_{\gamma-1}(\varepsilon)u^\varepsilon]_2(x, t) &= -2c_2^\varepsilon(1, x, t) \partial_\varepsilon^{\gamma-1} [\partial_\varepsilon (a^\varepsilon(1, t)r_1^\varepsilon(t)) [Gu^\varepsilon](t)] \\ &\quad - \int_1^x d_2^\varepsilon(\xi, x, t) [\tilde{F}_{\gamma-1}(\varepsilon)u^\varepsilon](\xi, \tau_2(\xi)) d\xi, \\ [\tilde{F}_{\gamma-1}(\varepsilon)u]_1(x, t) &= \left(-\partial_\varepsilon^{\gamma-1} [\partial_\varepsilon b_{11}^\varepsilon w_1^\varepsilon + \partial_\varepsilon b_{12}^\varepsilon w_2^\varepsilon + a_3 F(\varepsilon) u^\varepsilon], \right. \\ &\quad \left. -\partial_\varepsilon^{\gamma-1} [\partial_\varepsilon b_{21}^\varepsilon w_1^\varepsilon + \partial_\varepsilon b_{22}^\varepsilon w_2^\varepsilon + a_3 F(\varepsilon) u^\varepsilon] \right). \end{aligned}$$

First show that, given $\varepsilon \leq \varepsilon_0$, the equation (154) is well-defined in $\mathcal{C}^{k-\gamma}$. By Lemma 5.1 (iii), the operator $I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)$ is bijective from $\mathcal{C}^{k-\gamma}$ to $\mathcal{C}^{k-\gamma}$. Hence, it remains to

show that $\left[Q_{\gamma-1}(\varepsilon) + \tilde{R}_{\gamma-1}(\varepsilon)\right] u^\varepsilon \in \mathcal{C}^{k-\gamma}$. The induction assumption on γ implies that the map

$$\varepsilon \in [0, \varepsilon_0) \mapsto \tilde{R}_{\gamma-1}(\varepsilon) u^\varepsilon \in \mathcal{C}^{k-\gamma} \text{ is locally Lipschitz continuous.} \quad (155)$$

In particular, given $\varepsilon \leq \varepsilon_0$, $\left[\tilde{R}_{\gamma-1}(\varepsilon) u^\varepsilon\right](x, t) \in \mathcal{C}^{k-\gamma}$ as desired. In order to prove that $Q_{\gamma-1}(\varepsilon) u^\varepsilon \in \mathcal{C}^{k-\gamma}$, it is sufficient to show that $\partial_\varepsilon^{\gamma-1} \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma}$ or, the same, that $\partial_\varepsilon^{\gamma-1} \partial_t u^\varepsilon \in \mathcal{C}^{k-\gamma}$. For $\gamma = 1$ the statement is obvious. Using the induction argument, assume that $\partial_\varepsilon^{\gamma-2} \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma+1}$ or, the same, that $\partial_\varepsilon^{\gamma-2} \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma+1}$ for an arbitrary fixed $2 \leq \gamma \leq k-1$ and prove that $\partial_\varepsilon^{\gamma-1} \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma}$. For $w^\varepsilon = \partial_\varepsilon^{\gamma-1} u^\varepsilon$ we have the system (151)–(153) or the system (154) with γ replaced by $\gamma-1$. Using the induction assumption that $\partial_\varepsilon^{\gamma-2} \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma+1}$ and Lemma 5.1 (iii), we get $\partial_\varepsilon^{\gamma-1} u^\varepsilon \in \mathcal{C}^{k-\gamma+1}$. Hence, $\partial_\varepsilon^{\gamma-1} \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma}$ as desired. Consequently, the equation (154) determines uniquely w^ε as an element of $\mathcal{C}^{k-\gamma}$.

Further we state that w^ε given by (154) is locally Lipschitz continuous in $\mathcal{C}^{k-\gamma-1}$. Indeed, due to the induction assumption on γ the map $\varepsilon \in [0, \varepsilon_0) \mapsto u^\varepsilon \in \mathcal{C}^{k-\gamma}$ is $\mathcal{C}^{\gamma-1}$ -smooth in ε , hence, the map $\varepsilon \in [0, \varepsilon_0) \mapsto \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma-1}$ is $\mathcal{C}^{\gamma-1}$ -smooth in ε , from what follows that the map $\varepsilon \in [0, \varepsilon_0) \mapsto \partial_\varepsilon^{\gamma-1} \partial_x u^\varepsilon \in \mathcal{C}^{k-\gamma-1}$ is continuous in ε . Therefore, the map

$$\varepsilon \in [0, \varepsilon_0) \mapsto Q(\varepsilon) u^\varepsilon \in \mathcal{C}^{k-\gamma-1} \text{ is locally Lipschitz continuous.} \quad (156)$$

Combining statements (155) and (156) with Lemma 5.2 (ii) leads to the desired statement that w^ε is locally Lipschitz continuous in $\mathcal{C}^{k-\gamma-1}$.

Finally, we show that w^ε determined by (154) is in fact $\partial_\varepsilon^\gamma u^\varepsilon$. To this end, we adopt the convention that $Q_{-1}(\varepsilon) + \tilde{R}_{-1}(\varepsilon) = R(\varepsilon)$. Let us consider (154) with $\gamma-1$ in place of γ at some $\varepsilon \leq \varepsilon_0$ and $\varepsilon' \leq \varepsilon_0$. We thus have the following equalities in $\mathcal{C}^{k-\gamma-1}$:

$$\begin{aligned} \partial_\varepsilon^{\gamma-1} u^\varepsilon &= [B(\varepsilon) + A(\varepsilon) + D(\varepsilon)] \partial_\varepsilon^{\gamma-1} u^\varepsilon + Q_{\gamma-2}(\varepsilon) u^\varepsilon + \tilde{R}_{\gamma-2}(\varepsilon) u^\varepsilon, \\ \partial_\varepsilon^{\gamma-1} u^{\varepsilon'} &= [B(\varepsilon') + A(\varepsilon') + D(\varepsilon')] \partial_\varepsilon^{\gamma-1} u^{\varepsilon'} + Q_{\gamma-2}(\varepsilon') u^{\varepsilon'} + \tilde{R}_{\gamma-2}(\varepsilon') u^{\varepsilon'}. \end{aligned} \quad (157)$$

Using (154) and (157), we get

$$\begin{aligned} &[I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)] \left[\partial_\varepsilon^{\gamma-1} u^{\varepsilon'} - \partial_\varepsilon^{\gamma-1} u^\varepsilon - w^\varepsilon(\varepsilon' - \varepsilon) \right] \\ &= [I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)] \left[\partial_\varepsilon^{\gamma-1} u^{\varepsilon'} - \partial_\varepsilon^{\gamma-1} u^\varepsilon \right. \\ &\quad \left. - (\varepsilon' - \varepsilon) [I - B(\varepsilon) - A(\varepsilon) - D(\varepsilon)]^{-1} \left(Q_{\gamma-1}(\varepsilon) + \tilde{R}_{\gamma-1}(\varepsilon) \right) u^\varepsilon \right] \\ &= [(B(\varepsilon) - B(\varepsilon')) + (A(\varepsilon) - A(\varepsilon')) + (D(\varepsilon) - D(\varepsilon'))] \partial_\varepsilon^{\gamma-1} u^{\varepsilon'} \\ &\quad + \left[Q_{\gamma-2}(\varepsilon) + \tilde{R}_{\gamma-2}(\varepsilon) \right] (u^\varepsilon - u^{\varepsilon'}) + (\varepsilon' - \varepsilon) \left[(Q_{\gamma-2}(\varepsilon) - Q_{\gamma-2}(\varepsilon')) \right. \\ &\quad \left. + \left(\tilde{R}_{\gamma-2}(\varepsilon) - \tilde{R}_{\gamma-2}(\varepsilon') \right) - \left(Q_{\gamma-1}(\varepsilon) + \tilde{R}_{\gamma-1}(\varepsilon) \right) \right] u^\varepsilon \end{aligned}$$

The first summand is $o(|\varepsilon' - \varepsilon|)$ in $\mathcal{C}^{k-\gamma-1}$, as the maps $\varepsilon \in [0, \varepsilon_0) \mapsto B(\varepsilon)z \in \mathcal{C}^{k-\gamma-1}$, $\varepsilon \in [0, \varepsilon_0) \mapsto A(\varepsilon)z \in \mathcal{C}^{k-\gamma-1}$, and $\varepsilon \in [0, \varepsilon_0) \mapsto D(\varepsilon)z \in \mathcal{C}^{k-\gamma-1}$ are locally Lipschitz

continuous for all $z \in \mathcal{C}^{k-\gamma}$. The second summand is $o(|\varepsilon' - \varepsilon|)$ in $\mathcal{C}^{k-\gamma-1}$, since the maps $\varepsilon \in [0, \varepsilon_0) \mapsto Q_{\gamma-2}(\mu)u^\varepsilon \in \mathcal{C}^{k-\gamma}$ and $\varepsilon \in [0, \varepsilon_0) \mapsto \tilde{R}_{\gamma-2}(\mu)u^\varepsilon \in \mathcal{C}^{k-\gamma}$ are C^1 -smooth for all $\mu \leq \varepsilon_0$ and $u^\varepsilon \in \mathcal{C}^k$. The third summand is $o(|\varepsilon' - \varepsilon|)$ in $\mathcal{C}^{k-\gamma-1}$, due to the maps $\varepsilon \in [0, \varepsilon_0) \mapsto Q_{\gamma-2}(\varepsilon)z \in \mathcal{C}^{k-\gamma}$ and $\varepsilon \in [0, \varepsilon_0) \mapsto \tilde{R}_{\gamma-2}(\varepsilon)z \in \mathcal{C}^{k-\gamma}$ are C^1 -smooth for all $\mu \leq \varepsilon_0$ and $z \in \mathcal{C}^k$ and due to the estimate

$$\|Q_{\gamma-1}(\varepsilon)z\|_{\mathcal{C}^{k-\gamma-1}} + \|\tilde{R}_{\gamma-1}(\varepsilon)z\|_{\mathcal{C}^{k-\gamma-1}} = O(\|z\|_{\mathcal{C}^{k-\gamma-1}}),$$

being uniform in $\varepsilon \leq \varepsilon_0$ and $z \in \mathcal{C}^k$.

The proof of Theorem 1.5 is therewith complete.

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